5.1 Vectors

A **vector** is any quantity that has both magnitude and direction. For example, a description of the wind in a weather report typically includes both the strength of the wind and the direction from which the wind is blowing, making wind a vector quantity.

Many important quantities in science can be described as vectors, especially in physics. For example, a force acting on an object is a vector, since it has both magnitude (the strength of the force) and direction (the direction of the push or pull). Other vector quantities in physics include momentum, impulse, angular momentum, torque, and the electric and magnetic field vectors.

In mathematics, vectors are indispensable tools in geometry, and are used extensively to provide geometric insight in subjects such as multivariable calculus and differential equations. Vectors and their generalizations are also one of the main objects of study in linear algebra.

Vectors as Arrows

We typically think of a vector as an arrow drawn in Euclidean space. A **two-dimensional vector** can be represented by an arrow drawn in the Euclidean plane, as shown in Figure 1. The magnitude of the vector is length of the arrow, i.e. the distance from the point A to the point B, while the direction of the vector is the direction that the arrow points.

▲ **Figure 1:** A two-dimensional vector.

Two vectors are considered equal as long as they have the same magnitude and the same direction. For example, the two vectors shown in Figure 2 are considered the same, even though they start and end at different points. When working with vectors, we can move a vector anywhere we want in the plane as long as we do not change its magnitude or direction.

▲ **Figure 2:** These two vectors are equal, since they have the same magnitude and the same direction.

Notation and Terminology

When discussing vectors, ordinary numerical quantities (i.e. real numbers) are referred to as **scalars**. Both vectors and scalars are used for algebra, and a variable may be either a **vector variable** or a **scalar variable**.

To help distinguish between these two types of quantities, vector variables will always be written in an upright bold font:
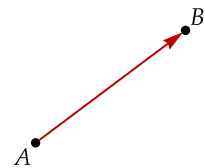
$$\mathbf{u}, \quad \mathbf{v}, \quad \text{or} \quad \mathbf{w},$$

The word "scalar" comes from the effect that scalars have on vectors when they multiply (see below).

while scalar variables will be written in the usual italic font:

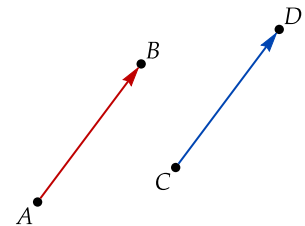$$x, \quad y, \quad \text{or} \quad z.$$

This is the most popular convention for distinguishing between vector and scalar variables, but other conventions are possible. For example, some books denote a vector variable using an arrow above the letter:

$$\vec{u}, \quad \vec{v}, \quad \text{or} \quad \vec{w}.$$

This is also the most common way to indicate a vector variable when doing hand calculations, since it is difficult to make handwritten letters bold, and the reader is encouraged to adopt this practice.
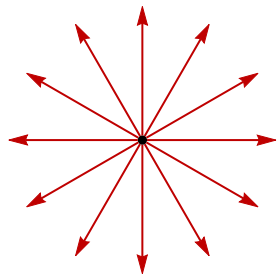
Advanced textbooks sometimes do not use any special notation to distinguish vector variables, especially in subjects that deal with many different kinds of mathematical objects. For such a book, it is up to the reader to remember which variables are scalars and which ones are vectors.

▲ These vectors all have the same magnitude but have different directions.

Magnitude of a Vector

The **magnitude** of a vector is a scalar, and corresponds to the length of any arrow that represents the vector. Different books write the magnitude of a vector **v** in different ways:

$$|\mathbf{v}|, \quad \|\mathbf{v}\|, \quad \text{or} \quad v.$$

Also, some books refer to the magnitude of a vector as its **length** or **norm** instead. We will consistently use the word "magnitude" and write $|\mathbf{v}|$ for the magnitude of a vector **v**.

The magnitude of a vector **v** is always either positive or zero:

$$\boxed{|\mathbf{v}| \geq 0}$$

If $|\mathbf{v}| = 0$, then **v** is the **zero vector**, which has a magnitude of zero but no direction. We will use a bold zero **0** to denote the zero vector.

Vector Addition
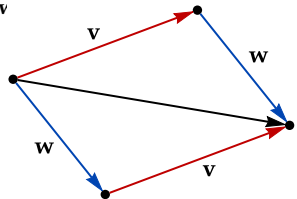
There is a way of combining two vectors known as **vector addition**. Given two vectors **v** and **w**, we arrange them end to end as shown in Figure 3. The vector that stretches from the beginning of **v** to the end of **w** is called the **sum** of **v** and **w**, and is written **v** + **w**.
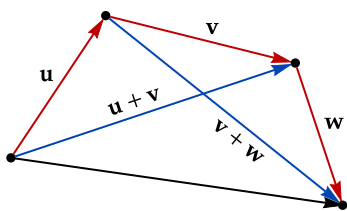
Vector addition has many of the same properties as addition of scalars.
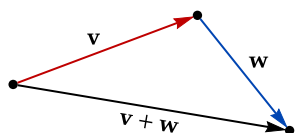
▲ **Figure 3:** The sum of two vectors **v** and **v**

▲ **Figure 4:** A demonstration that vector addition is commutative. The middle arrow represents both **v** + **w** and **w** + **v**

Properties of Vector Addition

Vector addition has the following properties:

1. Commutative property: **v** + **w** = **w** + **v** for any two vectors **v** and **w**.

2. Associative property: (**u** + **v**) + **w** = **u** + (**v** + **w**) for any three vectors **u**, **v**, **w**.

3. Additive identity: **v** + **0** = **v** for any vector **v**, where **0** is the zero vector.

Figure 4 shows the geometry behind the commutative property, and Figure 5 shows the geometry behind the associative property. The additive identity property follows immediately from the fact that the zero vector has the same starting and ending point.

▲ **Figure 5:** A demonstration that vector addition is associative. The bottom arrow represents both (**u** + **v**) + **w** and **u** + (**v** + **w**).

Negation and Subtraction

The **negation** of a vector **v**, denoted −**v**, is the vector with the same magnitude but the exact opposite direction, as shown in Figure 6. Sometimes −**v** is also called the **opposite** of **v**, and together **v** and −**v** are **opposite vectors**. Note that the negation of the zero vector is again the zero vector.

A vector and its negation are additive inverses, meaning that

$$\boxed{\mathbf{v} + (-\mathbf{v}) = \mathbf{0}}$$

for any vector **v**.

Although this operation is referred to as negation, there isn't any sense in which a vector is either positive or negative. A vector **v** points in a certain direction, and its negation −**v** points in the exact opposite direction, but neither of these directions is preferred, and there is no good way to decide which of **v** and −**v** should be positive and which should be negative.

We can use negation to define subtraction for vectors. If **v** and **w** are vectors, the **difference w** − **v** is defined by the formula
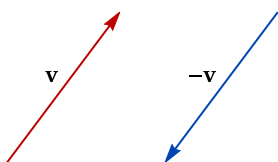
▲ **Figure 6:** A vector **v** and its negation −**v**

$$\mathbf{w} - \mathbf{v} \;=\; \mathbf{w} + (-\mathbf{v}).$$

Figure 7 shows the geometric meaning of vector subtraction. If $\mathbf{v}$ and $\mathbf{w}$ are arranged to have the same beginning point, then the difference $\mathbf{w} - \mathbf{v}$ is the vector that points from the end of $\mathbf{v}$ to the end of $\mathbf{w}$.

▲ **Figure 7:** The difference of two vectors $\mathbf{v}$ and $\mathbf{w}$.

Scalar Multiplication

There does not seem to be a good way of multiplying two vectors in general. However, there is an operation called **scalar multiplication** that allows us to multiply a vector by a scalar.

For a positive scalar k and a vector $\mathbf{v}$, the product $k\mathbf{v}$ is a vector in the same direction as $\mathbf{v}$ whose magnitude has been scaled by a factor of k. Thus,

$$|k\mathbf{v}| \;=\; k\,|\mathbf{v}|.$$

Figure 8 shows a vector $\mathbf{v}$ and several of its scalar multiples.

Multiplying a vector by negative scalar both scales and negates the vector. Thus $(-1)\mathbf{v}$ is the same as $-\mathbf{v}$, and $(-k)\mathbf{v}$ is the same as $-(k\mathbf{v})$ for any scalar k. Note then that

▲ **Figure 8:** A vector $\mathbf{v}$ and several of its scalar multiples.

$$|k\mathbf{v}| \;=\; |k|\,|\mathbf{v}|$$

for any scalar k, where $|k|$ is the absolute value of k.

Finally, multiplying a vector by the scalar 0 results in the zero vector:

$$0\mathbf{v} \;=\; \mathbf{0}$$

We now list some further properties of scalar multiplication. Most of these are quite straightforward, and the reader is encouraged to think about the justifications.

Properties of Scalar Multiplication

Scalar multiplication has the following properties:

1. Associativity: $a(b\mathbf{v}) = (ab)\mathbf{v}$ for any scalars a, b and any vector $\mathbf{v}$.

2. Multiplicative identity: $1\mathbf{v} = \mathbf{v}$ for any vector $\mathbf{v}$.

3. Left distributivity: $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$ for any scalar a and any vectors $\mathbf{v}, \mathbf{w}$.

4. Right distributivity: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ for any scalars a, b and any vector $\mathbf{v}$.
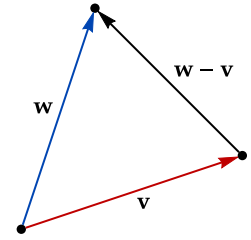

Linear Combinations

A **linear combination** of two vectors $\mathbf{v}$ and $\mathbf{w}$ is any expression of the form
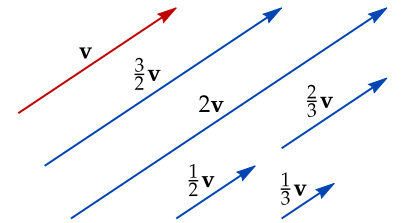
$$a\mathbf{v} + b\mathbf{w},$$

where a and b are scalars. Figure 9 shows two vectors $\mathbf{v}$ and $\mathbf{w}$ and several different linear combinations of them.
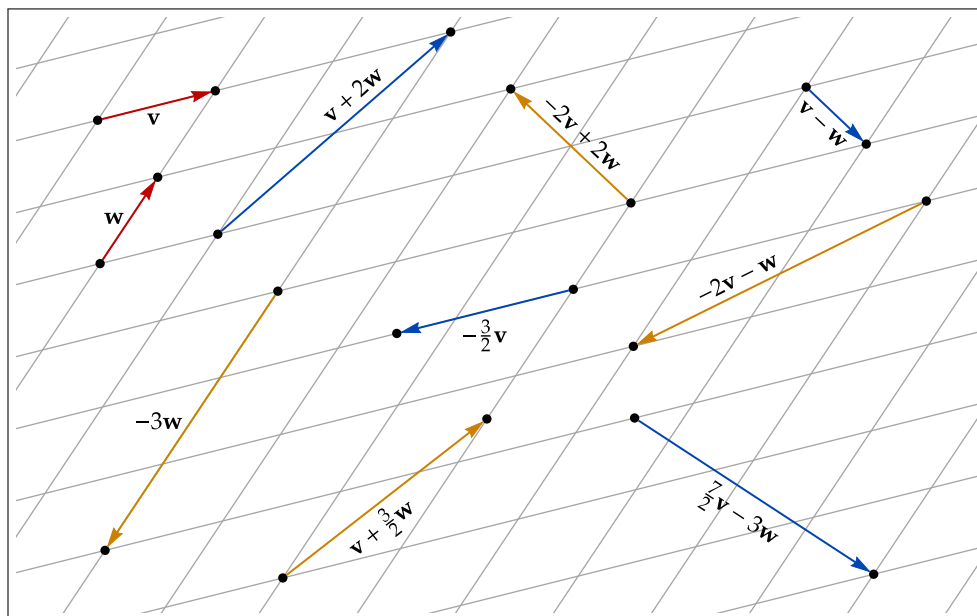
A linear combination is the most general way of combining two vectors $\mathbf{v}$ and $\mathbf{w}$ using vector addition and multiplication by scalars. That is, any expression involving $\mathbf{v}$ and $\mathbf{w}$ and these two operations can always be simplified algebraically to a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

► Figure 9: Several different linear combinations of two vectors **v** and **w**. Each parallelogram in the grid has **v** and **w** along its sides.

EXAMPLE 1
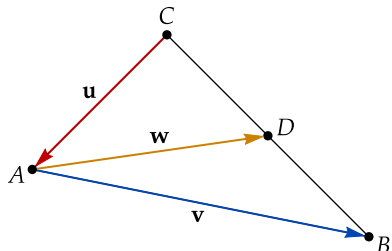
Express the vector $2(-3\mathbf{v} + 4\mathbf{w}) - 3(-5\mathbf{v})$ as a linear combination of **v** and **w**.

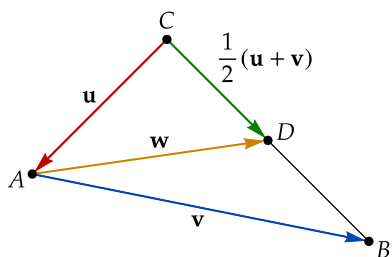SOLUTION   Distributing the 2 and -3 through gives

$$-6\mathbf{v} + 8\mathbf{w} + 15\mathbf{v},$$

and combining like terms yields the linear combination

$$9\mathbf{v} + 8\mathbf{w}.$$

▲ Figure 10: Diagram for Example 2.

EXAMPLE 2

In Figure 10, point D is the midpoint of line segment $\overline{BC}$. Express the vector **w** as a linear combination of **u** and **v**.

SOLUTION   Note first that the vector from C to B is equal to $\mathbf{u} + \mathbf{v}$. Since D is the midpoint of $\overline{BC}$, the vector from C to D must be half of $\mathbf{u} + \mathbf{v}$, as shown in Figure 11. From the picture, we see that

$$\mathbf{u} + \mathbf{w} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$$

and therefore

$$\mathbf{w} = -\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}.$$

▲ Figure 11: The vector from C to D is half of $\mathbf{u} + \mathbf{v}$.

It is also possible to take a linear combination of more than two vectors. For example, if **u**, **v**, and **w** are three vectors, a linear combination of them would have the form

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w},$$

where a, b, and c are scalars. More generally, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are n different vectors, then a linear combination of them has the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n,$$

where $a_1, a_2, \ldots, a_n$ are n scalars.

The Algebra of Vectors

The properties of vector addition and scalar multiplication mean that essentially all of the usual rules of algebra apply to vectors, as long we do not try to multiply two vectors together or add a vector to a scalar. In particular, we can use algebra to solve equations involving vectors.

EXAMPLE 3

Solve the following equation for $\mathbf{v}$ in terms of $\mathbf{u}$:

$$3\mathbf{u} + 2\mathbf{v} = 5\mathbf{u} - 4(\mathbf{u} - 2\mathbf{v}).$$

SOLUTION   Distributing the -4 on the right gives

$$3\mathbf{u} + 2\mathbf{v} = 5\mathbf{u} - 4\mathbf{u} + 8\mathbf{v},$$

which simplifies to

$$3\mathbf{u} + 2\mathbf{v} = \mathbf{u} + 8\mathbf{v}.$$

Subtracting both $8\mathbf{v}$ and $3\mathbf{u}$ from both sides gives

$$-6\mathbf{v} = -2\mathbf{u}$$

and dividing through by -6 yields

$$\mathbf{v} = \frac{1}{3}\mathbf{u}.$$

EXERCISES

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1. Two vectors $\mathbf{v}$ and $\mathbf{w}$ have magnitudes of 12 and 5, respectively.
   (a) What is the largest possible magnitude of $\mathbf{v} + \mathbf{w}$? What is the smallest possible magnitude?
   (b) Suppose the directions of $\mathbf{v}$ and $\mathbf{w}$ are perpendicular. What is the magnitude of $\mathbf{v} + \mathbf{w}$ in this case?

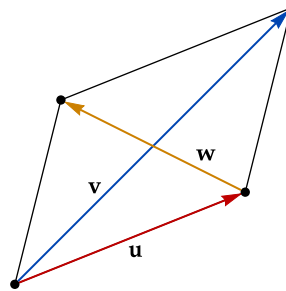2. The following figure shows two congruent rectangles.
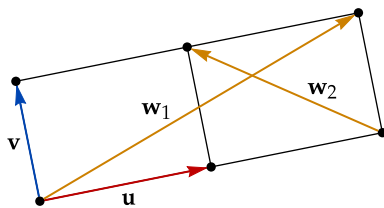
Express the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ as linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

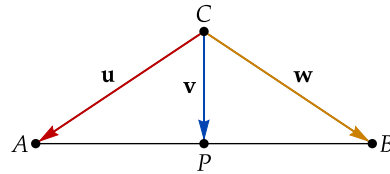**3.** The following figure shows a tiling of the plane by congruent hexagons.
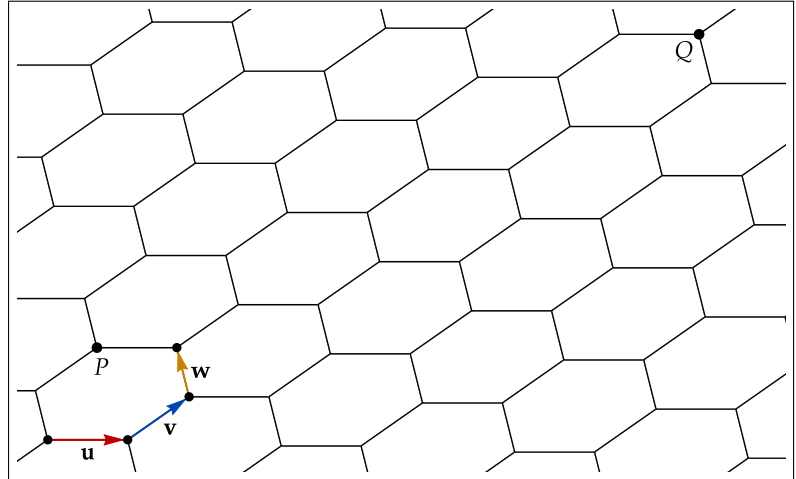
Express the vector from P to Q as a linear combination of **u**, **v**, and **w**.

4. The following figure shows a parallelogram in the plane.

Express the vector **w** as a linear combination of **u** and **v**.

5. In the following figure, $\triangle ABC$ is isosceles, and $\angle APC$ is a right angle.

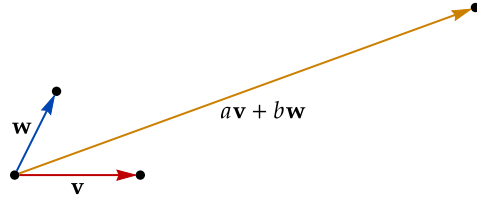Express the vector **w** as a linear combination of **u** and **v**.

6. The following figure shows a right triangle in the plane.
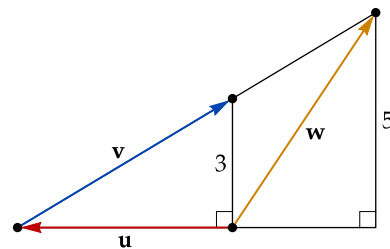
Express the vector **w** as a linear combination of **u** and **v**.

7. The following figure shows two vectors **v** and **w** as well as a certain linear combination $a\mathbf{v} + b\mathbf{w}$.

Use the picture to estimate the values of a and b. Try to be accurate to within 20%.

8. Given that $3(\mathbf{u} + 2\mathbf{w}) + 4\mathbf{v} = 2\mathbf{u} - 4(3\mathbf{v} - \mathbf{w})$, express the vector **w** as a linear combination of **u** and **v**.

5.2   Vector Components

So far all of our discussion of vectors has been from the viewpoint of Euclidean geometry. However, in modern mathematics it is much more common to take an analytic approach to geometry, identifying each point in the plane with an ordered pair (x, y) of real coordinates. The resulting plane is sometimes called the xy-**plane**, but in linear algebra and advanced mathematics it is more often referred to as $\mathbb{R}^2$.

The most important vectors in $\mathbb{R}^2$ are the **standard basis vectors i** and **j**, shown in Figure 1. The standard basis vectors both have a magnitude of 1, with **i** pointing in the positive x-direction and **j** pointing in the positive y-direction.

We can take linear combinations of **i** and **j** to get other vectors in the plane. For example, Figure 2 shows the vector $5\mathbf{i} + 3\mathbf{j}$, which stretches 5 units in the positive x-direction and 3 units in the positive y-direction. In fact, every vector **v** in $\mathbb{R}^2$ can be written as a linear combination of **i** and **j**:

$$\boxed{\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}}$$

The coefficients v_x and v_y in this linear combination are called the **components** of **v**. These represent how far the vector stretches in the x and y directions, as shown in Figure 3. In particular, v_x is the **horizontal component** of **v**, and v_y is the **vertical component** of **v**.

Note that each component may be positive or negative. For example, v_x is positive if **v** points generally to the right, and v_x is negative if **v** points generally to the left. Similarly, v_y is positive if **v** points generally upwards, and v_y is negative if **v** points generally downwards.

Column Vectors

There is another common notation for linear combinations of **i** and **j**. Given any scalars a and b, we let

$$\boxed{\begin{bmatrix} a \\ b \end{bmatrix} = a\,\mathbf{i} + b\,\mathbf{j}.}$$

That is, $\begin{bmatrix} a \\ b \end{bmatrix}$ is a vector whose components are a and b. A vector $\begin{bmatrix} a \\ b \end{bmatrix}$ written in this fashion is known as a **column vector**. This way of writing vectors is very popular within mathematics, but physicists usually prefer to write **i** and **j** explicitly.

Of course, any vector **v** can be written as a column vector, with

$$\boxed{\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}}$$

Figure 4 shows four different vectors in $\mathbb{R}^2$, written as column vectors.

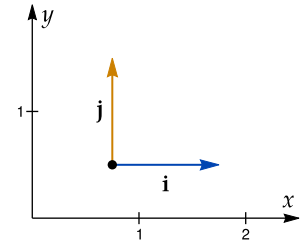Note that the standard basis vectors **i** and **j** can themselves be written as column vectors. In particular, since

$$\mathbf{i} = 1\mathbf{i} + 0\mathbf{j} \qquad \text{and} \qquad \mathbf{j} = 0\mathbf{i} + 1\mathbf{j}$$
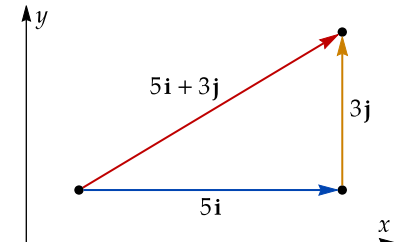
we have

$$\boxed{\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \qquad \text{and} \qquad \boxed{\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$
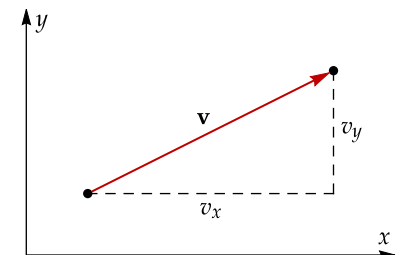
Here $\mathbb{R}$ is the symbol for the set of real numbers. The superscript 2 in $\mathbb{R}^2$ represents the fact that every point in the plane is an ordered *pair* of numbers.
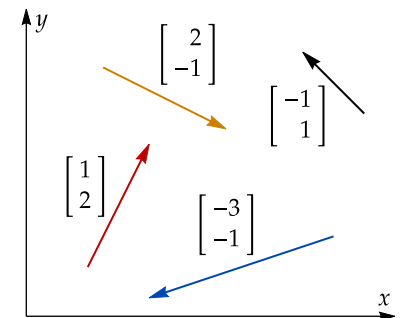

▲ **Figure 1:** The standard basis vectors **i** and **j**.

▲ **Figure 2:** A linear combination of **i** and **j**.

▲ **Figure 3:** The two components v_x and v_y of a vector **v**.

▲ **Figure 4:** Four vectors in $\mathbb{R}^2$.

Vector Operations

The operations of vector addition and scalar multiplication are easy to compute using components. The formula for vector addition is

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$$

That is, each component of the sum of two vectors is the sum of the corresponding components of the two vectors. For example,

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

The formula for scalar multiplication is equally simple:

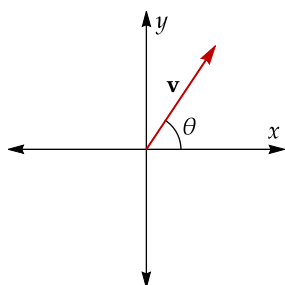$$k \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ kb \end{bmatrix}$$

That is, multiplying a vector by a scalar k is the same as multiplying each component by k. For example,

$$2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}.$$

Magnitude and Direction

▲ Figure 5: The direction of **v** is determined by the angle θ.

The direction of a vector **v** in $\mathbb{R}^2$ can be described by a single angle θ, as shown in Figure 5. As is the custom in mathematics, we measure angles counterclockwise from the positive x-axis. Thus:

- $\theta = 0°$ if the vector points to the right,

- $\theta = 90°$ if the vector points straight up,

- $\theta = 180°$ if the vector points to the left, and

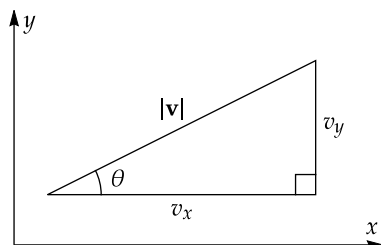- $\theta = 270°$ or $(-90°)$ if the vector points straight down.

Given the direction θ and magnitude $|\mathbf{v}|$ for a vector, we can find the components of the vector using the formulas

$$v_x = |\mathbf{v}| \cos \theta \quad \text{and} \quad v_y = |\mathbf{v}| \sin \theta$$

▲ Figure 6: The components of **v** are the legs of a right triangle whose hypotenuse is $|\mathbf{v}|$.

These formulas can be derived from the right triangle shown in Figure 6. Conversely, given the components of a vector, we can find the magnitude and direction using the formulas

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \quad \text{and} \quad \tan \theta = \frac{v_y}{v_x}$$

The first of these formulas follows from the Pythagorean theorem. The second allows us to find the angle θ using v_x and v_y, though we must be careful about the direction that the vector points.

⚠ WARNING The formula given for $\tan \theta$ does *not* imply that

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right).$$

This formula works if the vector points to the right (i.e. if $v_x > 0$), but if the vector points to the left the value of the arctangent will be off by $180°$.

EXAMPLE 1

Find the magnitude and direction of the vector $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

SOLUTION   From the formulas above, we have

$$|\mathbf{v}| = \sqrt{(-4)^2 + (3)^2} = 5 \qquad \text{and} \qquad \tan \theta = -\frac{3}{4},$$

where θ is the direction of $\mathbf{v}$. Now,

$$\tan^{-1}\left(-\frac{3}{4}\right) = -36.87°,$$

but this is not the right direction, since the vector would have to point down and to the right. Adding $180°$ gives the correct direction of $143.13°$.

Vector Equations

Since the components of a vector determine both the magnitude and direction, two vectors are equal if and only if they have the same components.

Equality of Vectors

The equation

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

holds if and only if $a = c$ and $b = d$.

Here **if and only if** means that either statement implies the other. That is, if two vectors are the equal then their components are the same, and if two vectors have the same components then they must be equal.

A **vector equation** is an equation for which both sides are vectors. The principle above lets us separate a vector equation into two scalar equations, which is often necessary for finding the solution.

EXAMPLE 2

Solve for x:

$$3\begin{bmatrix} x \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ -6 \end{bmatrix} = x\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

SOLUTION   Multiplying the left vector through by 3 and the right vector through by x gives the equation

$$\begin{bmatrix} 3x \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 2x \\ 3x \end{bmatrix}.$$

We can now subtract the two vectors on the left to get

$$\begin{bmatrix} 3x - 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2x \\ 3x \end{bmatrix},$$

which is the same as the two scalar equations

$$3x - 4 = 2x \qquad \text{and} \qquad 12 = 3x.$$

The solution to both of these equations is $x = 4$.

EXAMPLE 3

Solve for x and y:

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

SOLUTION   Combining the vectors on the left together gives the equation

$$\begin{bmatrix} 2x + 2y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix},$$

which is the same as the system of equations

$$2x + 2y = 8 \qquad \text{and} \qquad 3x + y = 6.$$

The second equation tells us that $y = 6 - 3x$. Plugging this into the first equation and solving for x yields $x = 1$, and it follows that $y = 3$.
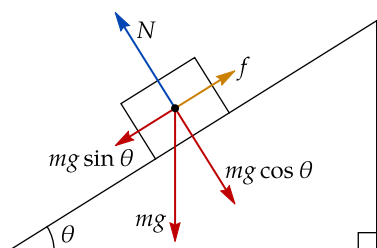

Vectors in Science

In science, vectors are used to represent any quantity that has both magnitude and direction. For example:

1. Any moving object has an associated **velocity vector**. The magnitude of this vector is the speed of motion, and the direction is the direction in which the object is moving.

2. In physics, a force acting on an object is represented by a **force vector**. The magnitude of this vector is the strength of the force, and the direction is the direction in which the force pushes.

▲ **Figure 7:** A free body diagram in physics represents force vectors as arrows.

Note that the magnitude of a vector may have any units, in which case the components of a vector have those same units. For example, the magnitude and components of a velocity vector have units of m/sec, while the magnitude and components of a force vector are measured in Newtons (N).

Vectors in science are often drawn as arrows, even if the vectors themselves do not have units of length. For example, a free body diagram in physics represents the forces acting on an object as arrows, as shown in Figure 7. In this case, the scale of the arrows is arbitrary, as long as all force vectors are drawn to the same scale.

Vector addition is very important for force vectors in physics. For a given object, the sum of all of the force vectors acting on it is called the **net force**, and determines the motion of the object. An object is in **static equilibrium** if it is at rest and the net force on it is equal to zero.

EXERCISES

1. Compute $2\mathbf{v} + 3\mathbf{w}$ if $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2. Find the magnitude and direction of the given vector, correct two two decimal places.

(a) $\begin{bmatrix} -6 \\ -8 \end{bmatrix}$ (b) $4\mathbf{i} - \mathbf{j}$ (c) $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

3. A vector $\mathbf{v}$ has a direction of $60°$ and $v_y = 6$. What is the magnitude of $\mathbf{v}$?

4. Find a vector $\mathbf{v}$ such that $2\mathbf{v} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$.

5. Find the value of x for which $\begin{bmatrix} x \\ 1 \end{bmatrix} + x \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} x \\ 7 \end{bmatrix}$.

6. Find the values of x and y for which $x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 17 \end{bmatrix}$.

7. A javelin is thrown with a speed of 25 m/sec at an angle of $40°$ above the horizontal. Find the horizontal and vertical components of the javelin's velocity vector.

8. The following picture shows three forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ acting on an object in static equilibrium.
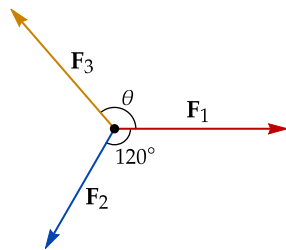
Given that $|\mathbf{F}_1| = 5.0$ N and $|\mathbf{F}_2| = 4.0$ N, find the magnitude and direction of the force $\mathbf{F}_3$.
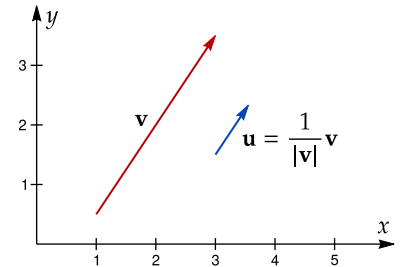
5.3 Vector Geometry

Unit Vectors

A vector $\mathbf{u}$ is called a **unit vector** if its magnitude is equal to 1. Such vectors are often used to represent a direction alone, without a corresponding magnitude. When writing unit vectors by hand, it is common to replace the arrow above the letter with a hat, i.e. $\hat{u}$ instead of $\vec{u}$. Some books also use hats for unit vectors in printed text, but we will just use plain bold letters.

Given any nonzero vector $\mathbf{v}$, we can find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ using the formula

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

▲ **Figure 1:** Dividing a vector by its magnitude produces a unit vector in the same direction.

Figure 1 shows a vector $\mathbf{v}$ and the resulting unit vector $\mathbf{u}$. The vector $\mathbf{u}$ is sometimes called the **normalization** of $\mathbf{v}$.

EXAMPLE 1

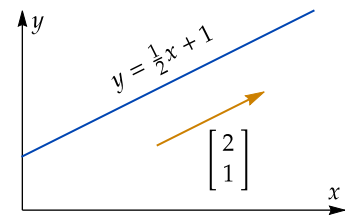Find a unit vector that is parallel to the line $y = \frac{1}{2}x + 1$.

SOLUTION   This line has slope $1/2$, so it is parallel to the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, as shown in Figure 2.
Since
$$|\mathbf{v}| = \sqrt{2^2 + 1^2} = \sqrt{5},$$
the desired unit vector is
$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

▲ **Figure 2:** The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is parallel to a line with slope $1/2$.

Any vector can be expressed as a scalar multiple of a unit vector. In particular, if $\mathbf{u}$ is a unit vector in the same direction as $\mathbf{v}$, then

$$\mathbf{v} = |\mathbf{v}|\mathbf{u}$$

This formula is often useful for building vectors with a certain magnitude and direction. The idea is to first find a unit vector $\mathbf{u}$ in the appropriate direction, and then multiply by the appropriate magnitude to obtain the desired vector.

EXAMPLE 2

Find a vector in the same direction as $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ that has magnitude 3.

SOLUTION   Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then $|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5$, so

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

is a unit vector in the same direction as $\mathbf{v}$. Multiplying $\mathbf{u}$ by 3 gives the desired vector:

$$3\mathbf{u} = \frac{3}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 12/5 \end{bmatrix}.$$

Points as Vectors

Given any point (x, y) in the plane, the associated **radial vector** is the vector that stretches from the origin $(0, 0)$ to the given point. The components of the radial vector are the same as the coordinates of the point:

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Figure 3: The point $(3, 2)$ and its associated radial vector.

For example, Figure 3 shows the point $(3, 2)$ and the corresponding radial vector.

Although we think of a point and its radial vector very differently, they are both described by the same pair of real numbers x, y. This makes it possible to regard points and vectors as the same thing. Indeed, **from now on we will make no distinction between a point in the plane and its radial vector**. Thus

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

Figure 4: The magnitude of a point is its distance from the origin.

for any real numbers x and y.

This convention simplifies our mathematics, because it decreases the number of different kinds of objects that we have to contend with. It also gives us the freedom to use vector operations such as addition and scalar multiplication on points as well as vectors. For example, we can now add two vectors, add two points, or add a vector to a point without worrying about the distinction between these operations.

Figure 5: The sum of two points $\mathbf{p}$ and $\mathbf{q}$.

Vector Operations Involving Points

We now have two different geometric interpretations of vectors. Given a vector such as $(3, 2)$, we can regard it as either an arrow whose components are 3 and 2, or as a point whose coordinates are 3 and 2. We already know the geometric meaning of vector operations involving arrows. But what do these operations look like for points?
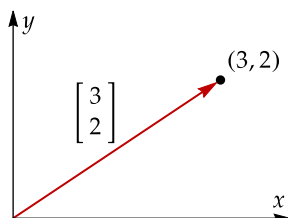
First observe that the magnitude of a point $\mathbf{p}$ is the same as the distance from $\mathbf{p}$ to the origin, as shown in Figure 4. Indeed, the formula for magnitude is the same as the familiar distance formula:
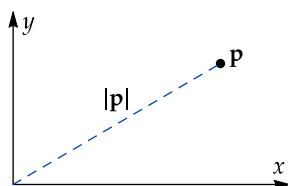
$$|(x, y)| = \sqrt{x^2 + y^2}.$$

Figure 6: The sum of a point $\mathbf{p}$ and a vector $\mathbf{v}$.

Next we consider the vector sum of two points, as shown in Figure 5. Given points $\mathbf{p}$ and $\mathbf{q}$, the sum $\mathbf{p} + \mathbf{q}$ is the fourth vertex of the parallelogram formed by the origin and the points $\mathbf{p}$ and $\mathbf{q}$. This rule for adding vectors is sometimes known as the **parallelogram law**.

There is also a geometric interpretation for adding a point to an arrow, as shown in Figure 6. Given a point $\mathbf{p}$ and a vector $\mathbf{v}$, if we move the arrow for $\mathbf{v}$ so that it begins at $\mathbf{p}$, then the other end of the arrow will be at $\mathbf{p} + \mathbf{v}$.

Figure 7: The difference of two points $\mathbf{p}$ and $\mathbf{q}$.
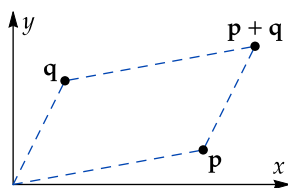
Similarly, if $\mathbf{p}$ and $\mathbf{q}$ are two points in the plane, the difference $\mathbf{q} - \mathbf{p}$ can be interpreted as the vector that goes from $\mathbf{p}$ to $\mathbf{q}$, as shown in Figure 7. This vector $\mathbf{q} - \mathbf{p}$ is sometimes called the **displacement** from the point $\mathbf{p}$ to the point $\mathbf{q}$.
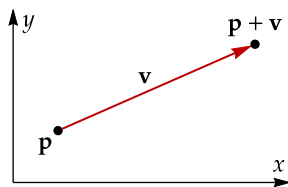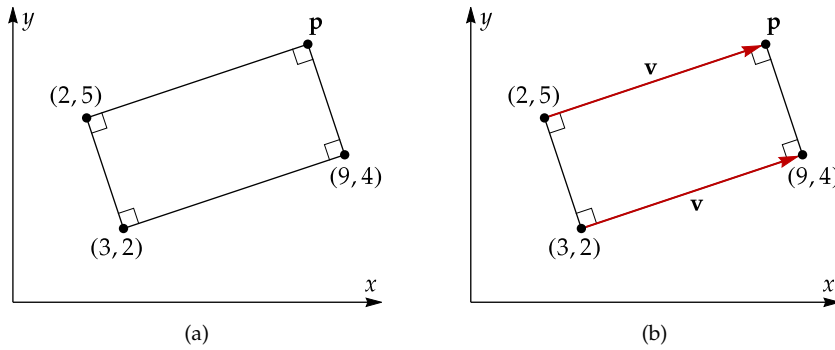
Figure 8: (a) A rectangle in the plane. (b) The vectors along the top and bottom are the same.

EXAMPLE 3

A rectangle in the plane has vertices at $(3,2)$, $(9,4)$, and $(2,5)$, a shown in Figure 8(a). Find the coordinates of the fourth vertex **p**.

SOLUTION   Let **v** be the vector that appears along both the top and bottom edges of the rectangle, as shown in Figure 8(b). Using the bottom edge, we have

$$\mathbf{v} = (9,4) - (3,2) = (6,2).$$

Then

$$\mathbf{p} = (2,5) + \mathbf{v} = (2,5) + (6,2) = (8,7).$$

As for scalar multiplication, Figure 9 shows several scalar multiples of a point **p**. Note that all of the scalar multiples of **p** lie on a line, namely the line that goes through the origin and the point **p**.

Certain linear combinations of **p** and **q** also have nice geometric interpretations. For example, the linear combination

$$\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$

can be interpreted as the midpoint of the line segment between **p** and **q**, as shown in Figure 10.

Figure 9: Several scalar multiples of a point **p**.

Figure 10: The midpoint between two points **p** and **q** is $\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$.

Turning Vectors 90°

Many geometry problems in the plane involve two perpendicular directions, so it can be helpful to know the rule for turning a vector 90°. Figure 11 shows the effect of such a turn on the components of a vector. As you can see, **you can turn a vector 90° by switching the two components and negating one of them**.
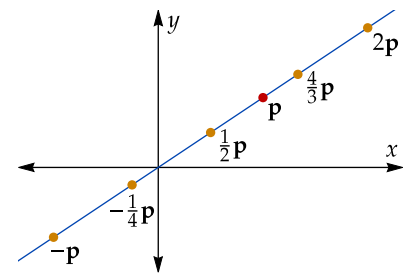
Of course, the direction in which you turn the vector depends on which of the two components you negate. Specifically, switching the two components and then negating the first component turns a vector 90° counterclockwise, while switching the two components and then negating the second component turns a vector 90° clockwise.
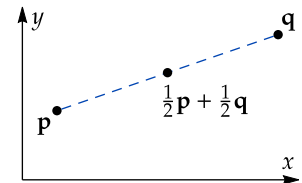
EXAMPLE 4

Figure 12(a) shows a square in the plane. Find the coordinates of the point **p**.

SOLUTION   Let **v** and **w** be the vectors shown in Figure 12(b). Then

$$\mathbf{v} = (6,2) - (2,1) = (4,1).$$

Figure 11: A vector can be turned 90° by switching its two components and negating one of them.

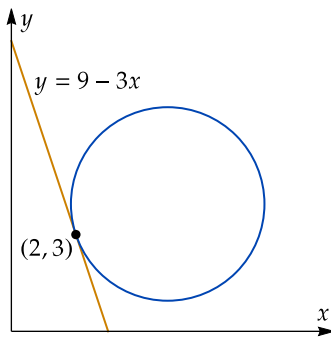**Figure 12:** (a) The square from Example 4. (b) The vectors **v** and **w** used in the solution to Example 4.
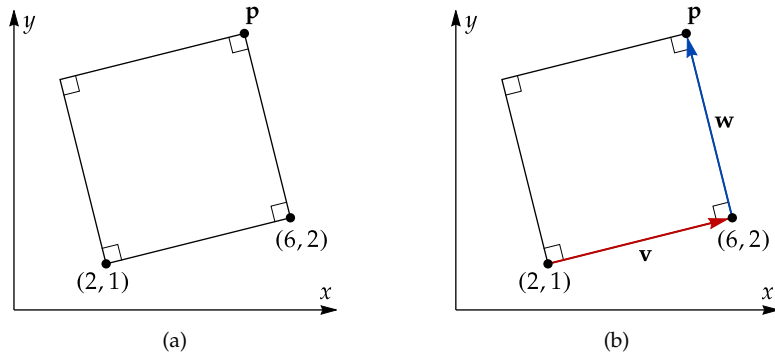
The vector **w** has the same length as **v**, but is turned 90° counterclockwise. We conclude that
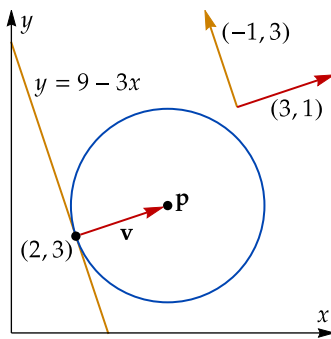
$$\mathbf{w} = (-1, 4),$$

so

$$\mathbf{p} = (6, 2) + \mathbf{w} = (6, 2) + (-1, 4) = (5, 6).$$

▲ **Figure 13:** The line and circle for Example 5.

▲ **Figure 14:** Finding the vector **v**.

EXAMPLE 5

A circle of radius 2 is tangent to the line $y = 9 - 3x$ at the point $(2, 3)$, as shown in Figure 13. Find the coordinates of the center of the circle.

SOLUTION   We wish to find the vector **v** that stretches from the point $(2, 3)$ to the center of the circle, as shown in Figure 14. We know that the magnitude of **v** is 2, so we need only find its direction.

The direction of **v** is perpendicular to that of the line $y = 9 - 3x$. This line has slope -3, and is therefore parallel to the vector $(-1, 3)$. Turning this vector 90° clockwise, we conclude that **v** is parallel to the vector $(3, 1)$.

All that remains is to scale the vector $(3, 1)$ to have length 2. Since

$$|(3, 1)| = \sqrt{3^2 + 1^2} = \sqrt{10}.$$

the vector

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is a unit vector in the direction of **v**. Since $|\mathbf{v}| = 2$, we conclude that

$$\mathbf{v} = \frac{2}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then

$$\mathbf{p} = (2, 3) + \mathbf{v} = (2, 3) + \frac{2}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \left(2 + \frac{6}{\sqrt{10}}, \, 3 + \frac{2}{\sqrt{10}} \right).$$

EXERCISES

1. Find a unit vector that points in the same direction as the vector $(-1, 1)$.

2. Find a vector with magnitude 5 that is parallel to the line $y = 2x + 1$.

3. Find a unit vector that points at an angle of $30°$ from the horizontal.

4. Find a unit vector whose direction is perpendicular to the line $y = \dfrac{2}{3}x$.

5. The following figure shows a parallelogram in the plane. Find the coordinates of the point **p**.
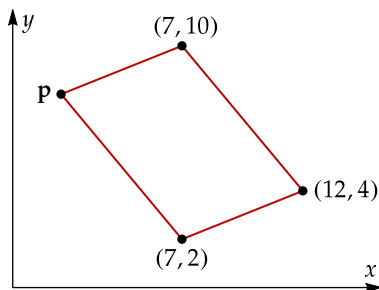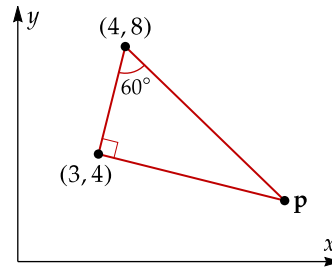

6. The two line segments in the following figure are congruent, perpendicular, and bisect one another. Find the coordinates of the point **p**.
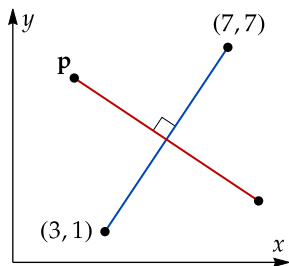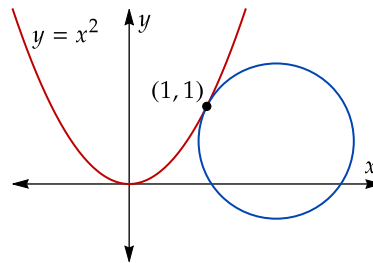

7. In the following figure, a rectangle with a length of 10 and a width of 5 is resting on the line $y = \dfrac{3}{4}x$. Find the coordinates of the point **p**.
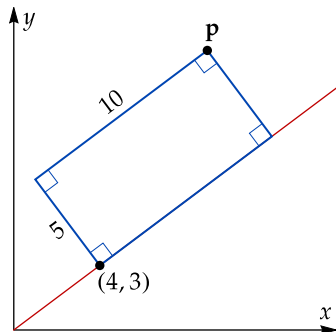
8. The following figure shows a right triangle in the plane. Find the coordinates of the point **p**.

9. In the following figure, a circle with a radius of 1 is tangent to the parabola $y = x^2$ at the point $(1, 1)$.

Find the coordinates of the center of the circle. (*Hint:* What is the slope of the tangent line?)