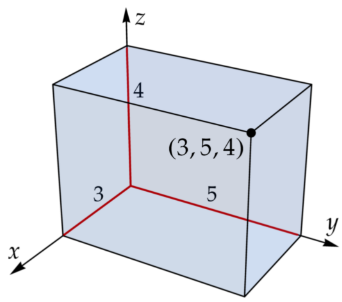
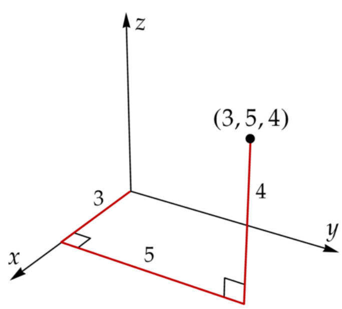
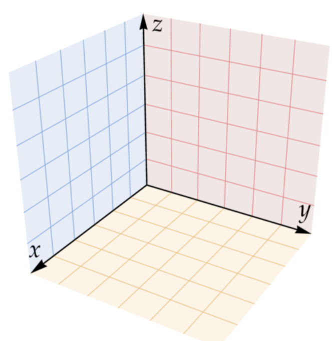
6.1 Three-Dimensional Space



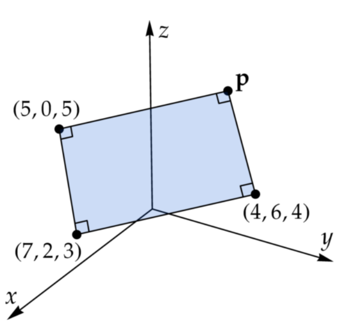
▲ **Figure 1:** Each point in three-dimensional space has three coordinates.



▲ **Figure 2:** A slightly different picture of the three coordinates.



▲ **Figure 3:** The three coordinate planes. The xy-plane (orange) is horizontal, while the xz-plane (blue) and the yz-plane (red) are vertical.



▲ **Figure 4:** A rectangle in $\mathbb{R}^3$.

So far we have only considered vectors and vector geometry in two-dimensional space. In this section we expand our point of view to consider three-dimensional space.

Three-Dimensional Coordinates

Just as a point on the plane can be described by two coordinates x and y, a point in three-dimensional space can be described by three coordinates x, y, and z. The geometric meaning of these coordinates is shown in Figures 1 and 2.

Using these coordinates, we can identify the points in three-dimensional space with **ordered triples** (x, y, z) of real numbers. In linear algebra and advanced mathematics, the set of all such ordered triples is usually denoted $\mathbb{R}^3$ (pronounced "arr-three").

The space $\mathbb{R}^3$ has three **coordinates axes**, namely the x-axis, the y-axis, and z-axis. These three axes are all mutually perpendicular, and they meet at the origin $(0, 0, 0)$. We usually think of the x and y axes as horizontal, while the z-axis is vertical.

There are also three **coordinate planes**, namely the xy-plane, the xz-plane, and the yz-plane, as shown in Figure 3. Each of these planes contains two of the coordinate axes, and is perpendicular to the third. These planes partition $\mathbb{R}^3$ into eight **octants**, similar to the four quadrants in $\mathbb{R}^2$. Though there is not a standard numbering for these octants, everyone agrees that the **first octant** is the portion of $\mathbb{R}^3$ consisting of all points whose coordinates are positive, i.e. the visible portion in Figure 3.

Vectors in $\mathbb{R}^3$

Every vector $\mathbf{v}$ in $\mathbb{R}^3$ has three components v_x, v_y, and v_z. As in $\mathbb{R}^2$, we think of vectors and points as the same thing, so

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = (v_x, v_y, v_z) = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}.$$

Here $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the three **standard basis vectors** in $\mathbb{R}^3$:

$$\boxed{\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

Addition and scalar multiplication in $\mathbb{R}^3$ work the same way that they do in $\mathbb{R}^2$:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} a+a' \\ b+b' \\ c+c' \end{bmatrix} \qquad \text{and} \qquad k\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}.$$

EXAMPLE 1

Figure 4 shows a rectangle in $\mathbb{R}^3$. Find the coordinates of the point $\mathbf{p}$.

SOLUTION Let $\mathbf{v}$ be the vector from $(7, 2, 3)$ to $(4, 6, 4)$. Then

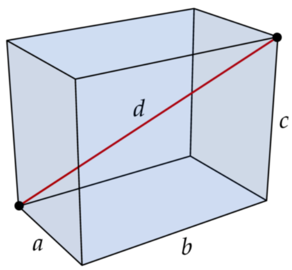
$$\mathbf{v} = (4, 6, 4) - (7, 2, 3) = (-3, 4, 1).$$

This same vector lies along the top edge of the rectangle from $(5, 0, 5)$ to $\mathbf{p}$, so

$$\mathbf{p} = (5, 0, 5) + \mathbf{v} = (5, 0, 5) + (-3, 4, 1) = (2, 4, 6).$$

Distance in $\mathbb{R}^3$

There is a nice formula for distances in $\mathbb{R}^3$, which is based on a three-dimensional version of the Pythagorean theorem.

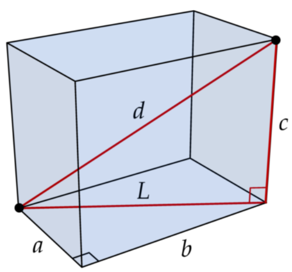


▲ **Figure 5:** A rectangular box.

Three-Dimensional Pythagorean Theorem

For a rectangular box with length a, width b, and height c, as shown in Figure 5, the distance d between opposite corners of the box obeys the formula

$$a^2 + b^2 + c^2 = d^2.$$



▲ **Figure 6:** Justification for the three-dimensional Pythagorean theorem.

This theorem can be justified using two applications of the usual Pythagorean theorem, as shown in Figure 6. The two right triangles in this figure give us the equations

$$a^2 + b^2 = L^2 \qquad \text{and} \qquad L^2 + c^2 = d^2,$$

and combining these gives the equation $a^2 + b^2 + c^2 = d^2$.

Based on this theorem, the magnitude of a vector $\mathbf{v}$ in $\mathbb{R}^3$ should be given by the formula

$$\boxed{|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}}$$

EXAMPLE 2

Find a vector parallel to $(2, 1, 2)$ that has a magnitude of 12.

SOLUTION We have

$$|(2, 1, 2)| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3,$$

so the vector $(2, 1, 2)$ has a magnitude of 3. To find a vector in the same direction with a magnitude of 12, we simply scale by a factor of 4:

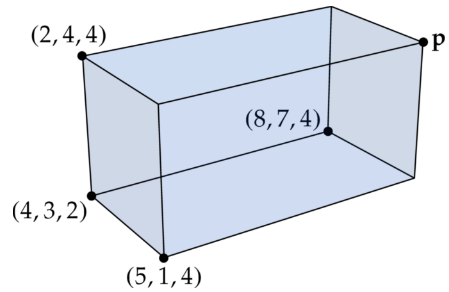
$$4(2, 1, 2) = \boxed{(8, 4, 8)}$$

EXERCISES

1. Find the magnitude of the vector $\begin{bmatrix} 8 \\ 1 \\ -4 \end{bmatrix}$.

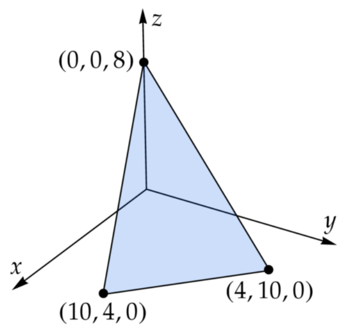
2. Find the distance between the points $(6, 3, 2)$ and $(2, 7, 9)$.

3. The following figure shows a rectangular box in $\mathbb{R}^3$.



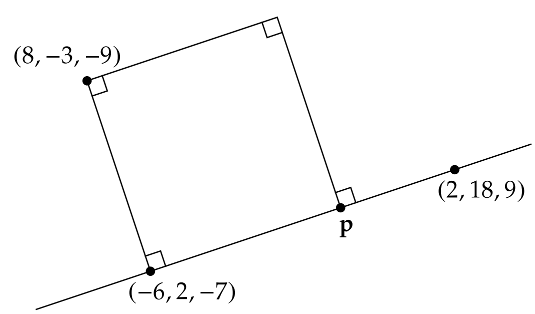
(a) Find the coordinates of the point $\mathbf{p}$.
(b) What is the volume of the box?

4. The following figure shows an isosceles triangle in $\mathbb{R}^3$.



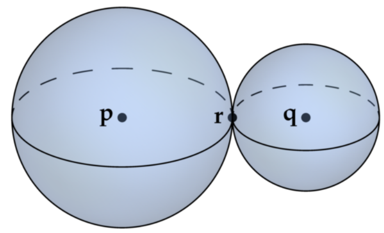
(a) Find the perimeter of the triangle.
(b) Find the area of the triangle.

5. The following figure shows a square and a line in $\mathbb{R}^3$.



Find the coordinates of the point $\mathbf{p}$.

6. The following figure shows two spheres in $\mathbb{R}^3$ that are tangent at a point $\mathbf{r}$.

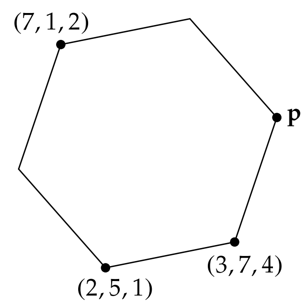


The larger sphere is centered at $\mathbf{p} = (-5, 0, 5)$, and has a radius of 9. The smaller sphere is centered at $\mathbf{q} = (5, 5, -5)$.

(a) What is the radius of the smaller sphere?

(b) Find the coordinates of the point $\mathbf{r}$ at which the two spheres are tangent.

7. The following figure shows a regular hexagon in $\mathbb{R}^3$.



Find the coordinates of the point $\mathbf{p}$.

6.2 Dot Product

The **dot product** is an operation that combines two vectors to obtain a scalar. For vectors in $\mathbb{R}^2$, it is defined by the formula

$$\boxed{\mathbf{v} \cdot \mathbf{w} \;=\; v_x w_x \;+\; v_y w_y}$$

That is, we multiply corresponding components of the two vectors and then add the results. For example,

$$(2,3) \cdot (5,4) \;=\; (2)(5) + (3)(4) \;=\; 10 + 12 \;=\; 22.$$

Because the result is a scalar, the dot product is also known as the **scalar product** of vectors. It is also sometimes called the **inner product** of vectors, or more properly the **Euclidean inner product**.

The dot product works just as well in $\mathbb{R}^3$:

$$\boxed{\mathbf{v} \cdot \mathbf{w} \;=\; v_x w_x \;+\; v_y w_y \;+\; v_z w_z}$$

For example,

$$(2,1,7) \cdot (3,4,-2) \;=\; (2)(3) + (1)(4) + (7)(-2) \;=\; -4.$$

Note that the dot product of these vectors is negative. In general, the dot product of two vectors may be positive, negative, or zero.

All of the discussion in this section applies equally well to vectors in two or three dimensions.

Algebraic Properties

The dot product can be thought of as a kind of multiplication for vectors. Indeed, the reason that we use a dot for dot product is that this is one of the symbols for multiplication. Many of the algebraic properties of multiplication are shared by dot product.

Algebraic Properties of the Dot Product

The dot product has the following properties:

1. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ for any two vectors $\mathbf{v}$ and $\mathbf{w}$.
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \;=\; \mathbf{u} \cdot \mathbf{v} \;+\; \mathbf{u} \cdot \mathbf{w}$ for any three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
3. $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{v} \cdot \mathbf{w})$ for any scalar k and any vectors $\mathbf{v}$ and $\mathbf{w}$.

Since dot product is commutative, it is also true that

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} \;=\; \mathbf{u} \cdot \mathbf{w} \;+\; \mathbf{v} \cdot \mathbf{w}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, and

$$\mathbf{v} \cdot (k\mathbf{w}) \;=\; k(\mathbf{v} \cdot \mathbf{w})$$

for any vectors $\mathbf{v}, \mathbf{w}$ and scalar k.

The first rule above is the **commutative law** for dot product, and the second is the **distributive law**. The third can be thought of as a version of the associative law, but it involves both dot product and scalar multiplication.

For multiplication of numbers, the **associative law** states that

$$(xy)z = x(yz)$$

for all real numbers x, y, z. The **commutative law** states that

$$xy = yx$$

for all real numbers x and y. Finally, the **distributive law** states that

$$x(y + z) = xy + xz$$

for all real numbers x, y, z.

Although the dot product shares many properties of multiplication, one should always keep in mind that the analogy between dot product and multiplication is not exact. For example, the product of two numbers is another number, which makes it possibly to multiply three numbers together. However, the dot product of two vectors is a scalar, not a vector, which makes it impossible to take the dot product of three or more vectors.

Incidentally, because the dot product obeys the distributive law, we can distribute dot products of sums just as we do in normal algebra. For example,

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) \;=\; \mathbf{a} \cdot \mathbf{c} \;+\; \mathbf{a} \cdot \mathbf{d} \;+\; \mathbf{b} \cdot \mathbf{c} \;+\; \mathbf{b} \cdot \mathbf{d}$$

for any four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

Dot Product Squares

Taking the dot product of a vector with itself gives the sum of the squares of the coordinates:

$$\mathbf{v} \cdot \mathbf{v} = v_x^2 + v_y^2 + v_z^2.$$

This is the same as the square of the magnitude of $\mathbf{v}$.

$$\boxed{\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2}$$

This relation between dot products and magnitudes of vectors can be useful for finding magnitudes of sums. For example, if $\mathbf{v}$ and $\mathbf{w}$ are vectors, the magnitude of $\mathbf{v}+\mathbf{w}$ obeys the formula

$$|\mathbf{v}+\mathbf{w}|^2 = (\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}+\mathbf{w}).$$

Distributing the right side gives

$$|\mathbf{v}+\mathbf{w}|^2 = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w}.$$

We can combine the middle terms since $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

Combining the two middle terms and replacing $\mathbf{v} \cdot \mathbf{v}$ and $\mathbf{w} \cdot \mathbf{w}$ by $|\mathbf{v}|^2$ and $|\mathbf{w}|^2$, we arrive at the formula

$$\boxed{|\mathbf{v}+\mathbf{w}|^2 = |\mathbf{v}|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + |\mathbf{w}|^2}$$

This is like a vector version of the formula $(a+b)^2 = a^2+2ab+b^2$. A similar computation for $\mathbf{v}-\mathbf{w}$ gives

$$\boxed{|\mathbf{v}-\mathbf{w}|^2 = |\mathbf{v}|^2 - 2(\mathbf{v} \cdot \mathbf{w}) + |\mathbf{w}|^2}$$

EXAMPLE 1

Find the magnitude of $\mathbf{v}+\mathbf{w}$ if $|\mathbf{v}| = 5$, $|\mathbf{w}| = 6$, and $\mathbf{v} \cdot \mathbf{w} = 10$.

SOLUTION We have

$$|\mathbf{v}+\mathbf{w}|^2 = |\mathbf{v}|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + |\mathbf{w}|^2 = (5)^2 + 2(10) + (6)^2 = 81$$

and therefore $|\mathbf{v}+\mathbf{w}| = 9$.

EXAMPLE 2

Find a formula for the magnitude of $2\mathbf{v}+3\mathbf{w}$ in terms of $|\mathbf{v}|$, $|\mathbf{w}|$, and $\mathbf{v} \cdot \mathbf{w}$.

SOLUTION We have

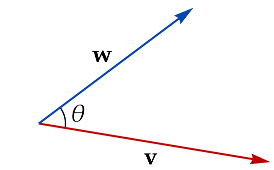
$$\begin{aligned} |2\mathbf{v}+3\mathbf{w}|^2 &= (2\mathbf{v}+3\mathbf{w}) \cdot (2\mathbf{v}+3\mathbf{w}) \\ &= 4(\mathbf{v} \cdot \mathbf{v}) + 6(\mathbf{v} \cdot \mathbf{w}) + 6(\mathbf{w} \cdot \mathbf{v}) + 9(\mathbf{w} \cdot \mathbf{w}) \\ &= 4|\mathbf{v}|^2 + 12(\mathbf{v} \cdot \mathbf{w}) + 9|\mathbf{w}|^2 \end{aligned}$$

and therefore

$$|2\mathbf{v}+3\mathbf{w}| = \sqrt{4|\mathbf{v}|^2 + 12(\mathbf{v} \cdot \mathbf{w}) + 9|\mathbf{w}|^2}$$

Geometric Interpretation

The importance of the dot product stems from its geometric interpretation.



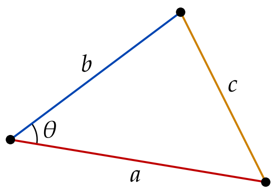
▲ **Figure 1:** The angle between $\mathbf{v}$ and $\mathbf{w}$.

Geometric Interpretation of Dot Product

If $\mathbf{v}$ and $\mathbf{w}$ are vectors, then

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, as shown in Figure 1.



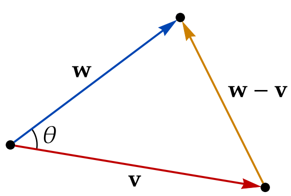
▲ **Figure 2:** Triangle for the law of cosines.

This formula follows from the **law of cosines** from trigonometry, which states that

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

for any triangle with side lengths a, b, c and angle θ as shown in Figure 2. If we apply the law of cosines to the triangle of vectors shown in Figure 3, we get the equation

$$|\mathbf{w} - \mathbf{v}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2|\mathbf{v}||\mathbf{w}|\cos\theta$$



▲ **Figure 3:** A triangle of vectors.

But we also know that

$$|\mathbf{w} - \mathbf{v}|^2 = |\mathbf{v}|^2 - 2(\mathbf{v} \cdot \mathbf{w}) + |\mathbf{w}|^2$$

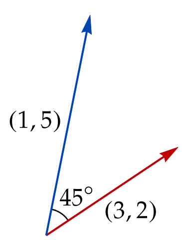
and it follows that $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$.

EXAMPLE 3

Two vectors $\mathbf{v}$ and $\mathbf{w}$ have magnitudes 5 and 6, and the angle between them is 60°. What is the value of $\mathbf{v} \cdot \mathbf{w}$?

SOLUTION We have $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta = (5)(6)\cos(60°) = (5)(6)(1/2) = 15$.

One way the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$ is useful is that it allows us to find the angle between two vectors.



▲ **Figure 4:** The angle between the vectors $(3, 2)$ and $(1, 5)$ is 45°.

EXAMPLE 4

Find the angle between the vectors $(3, 2)$ and $(1, 5)$.

SOLUTION Let $\mathbf{v} = (3, 2)$ and $\mathbf{w} = (1, 5)$. Then

$$\mathbf{v} \cdot \mathbf{w} = (3)(1) + (2)(5) = 13, \qquad |\mathbf{v}| = \sqrt{13}, \qquad \text{and} \qquad |\mathbf{w}| = \sqrt{26},$$

Substituting these into the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$ gives

$$13 = \sqrt{13}\sqrt{26}\cos\theta.$$

Then

$$\cos\theta = \frac{13}{\sqrt{13}\sqrt{26}} = \frac{1}{\sqrt{2}}$$

so $\theta = 45°$, as shown in Figure 4.

In general, the procedure is to solve for $\cos\theta$ and then take the inverse cosine, which gives an angle between 0° and 180°.

The Sign of the Dot Product

Another application of the formula

$$\mathbf{v} \cdot \mathbf{w} \;=\; |\mathbf{v}|\,|\mathbf{w}| \cos\theta$$

is that it gives us a general rule for the sign of $\mathbf{v} \cdot \mathbf{w}$. Assuming neither $\mathbf{v}$ nor $\mathbf{w}$ is the zero vector, the magnitudes $|\mathbf{v}|$ and $|\mathbf{w}|$ are always positive.dot product $\mathbf{v} \cdot \mathbf{w}$ always has the same sign as $\cos\theta$. Thus $\mathbf{v} \cdot \mathbf{w}$ can only be negative if $\cos\theta$ is negative.

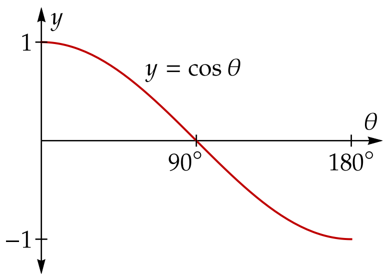


Figure 5: The cosine is positive for $\theta < 90°$ and negative for $\theta > 90°$.

Figure 5 shows the graph of the cosine function for $0° \leq \theta \leq 180°$. As you can see, the cosine is positive for acute angles (less than 90°) and negative for obtuse angles (greater than 90°). Thus:

- $\mathbf{v} \cdot \mathbf{w} > 0$ if the angle between $\mathbf{v}$ and $\mathbf{w}$ is acute.
- $\mathbf{v} \cdot \mathbf{w} = 0$ if the angle between $\mathbf{v}$ and $\mathbf{w}$ is a right angle.
- $\mathbf{v} \cdot \mathbf{w} < 0$ if the angle between $\mathbf{v}$ and $\mathbf{w}$ is obtuse.

This trichotomy is illustrated in Figure 6. The case where the dot product is zero has a special name.

Orthogonal Vectors

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$.

Note that the zero vector $\mathbf{0}$ is technically orthogonal to any other vector, since $\mathbf{0} \cdot \mathbf{v} = 0$ for any vector $\mathbf{v}$.

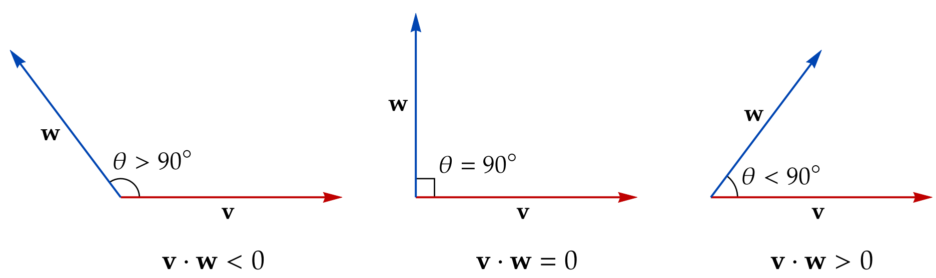
That is, two vectors are orthogonal if their directions are perpendicular. For example, the vectors $(5, 2)$ and $(-2, 5)$ are turned 90° from one another, so their dot product is equal zero:

$$(5,2) \cdot (-2,5) \;=\; (5)(-2) + (2)(5) \;=\; 0.$$

Orthogonal vectors in $\mathbb{R}^3$ tend to be much less obvious. For example, the vectors $(3, -5, 1)$ and $(7, 3, -6)$ are orthogonal in $\mathbb{R}^3$, since

$$(3,-5,1) \cdot (7,3,-6) \;=\; (3)(7) + (-5)(3) + (1)(-6) \;=\; 0.$$

Figure 6: The sign of the dot product depends on the angle between the two vectors.



EXERCISES

1. Compute the dot product of the vectors $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

2. Simplify $(\mathbf{v} + 3\mathbf{w}) \cdot \mathbf{v} \;+\; (3\mathbf{v}) \cdot (\mathbf{v} - \mathbf{w})$.

3. Find a formula for $|5\mathbf{v} - 3\mathbf{w}|$ in terms of $|\mathbf{v}|$, $|\mathbf{w}|$, and $\mathbf{v} \cdot \mathbf{w}$.

4. Find the magnitude of $\mathbf{v} + 2\mathbf{w}$ if $|\mathbf{v}| = 5$, $|\mathbf{w}| = 2$, and $\mathbf{v} \cdot \mathbf{w} = 2$.

5. Given that $|\mathbf{v}| = 7$ and $|\mathbf{w}| = 5$, find the value of $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w})$.

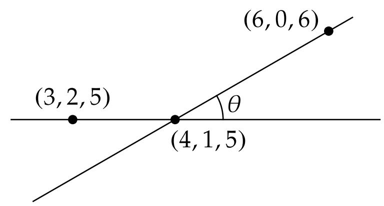
6. Suppose that two vectors $\mathbf{v}$ and $\mathbf{w}$ meet at a 30° angle. If $\mathbf{v} \cdot \mathbf{v} = 4$ and $\mathbf{w} \cdot \mathbf{w} = 12$, what is the value of $\mathbf{v} \cdot \mathbf{w}$?

7. Find the angle between the vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

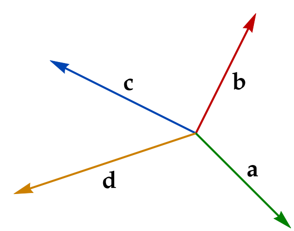
8. Compute the angle between the vectors $(2, 5, 1)$ and $(-1, 4, 2)$, correct to the nearest degree.

9. Find the angle between the vectors $\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $\mathbf{j} + \mathbf{k}$.

10. Find the angle θ between the two lines in the following figure.



11. The following figure shows four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ in $\mathbb{R}^2$.



(a) For which two of these vectors is the dot product positive?

(b) Which two of these vectors are orthogonal?

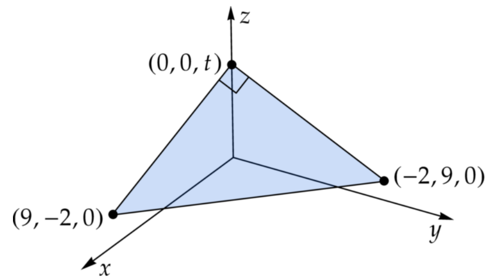
12. Let $\mathbf{a} = (-9, -3)$, $\mathbf{b} = (3, -8)$, $\mathbf{c} = (7, 2)$, and $\mathbf{d} = (-2, 6)$.

(a) Which two of these vectors are orthogonal?

(b) Which two of these vectors meet at an acute angle?

13. Find a value of t such that $(2, 3, t)$ is orthogonal to $5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

14. The following figure shows a right triangle in $\mathbb{R}^3$.



(a) Determine the value of t.

(b) What is the area of the triangle?