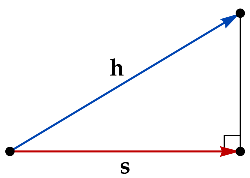
7.1 Projections and Components

As we have seen, the dot product of two vectors tells us the cosine of the angle between them. So far, we have only used this to find the angle between two vectors, but cosines are also useful for triangle trigonometry. In this section, we will learn how to use the dot product to derive information about right triangles.



▲ **Figure 1:** A right triangle with vectors **h** and **s**.

Projection Formula

Let T be a right triangle, let $\mathbf{h}$ be a vector along the hypotenuse, and let $\mathbf{s}$ be the vector emanating from the same vertex along the other side, as shown in Figure 1. Then

$$|\mathbf{s}| \;=\; \mathbf{h}\cdot\mathbf{u},$$

where $\mathbf{u}$ is a unit vector in the direction of $\mathbf{s}$.

The scalar $|\mathbf{s}|$ is sometimes called the **projection** of $\mathbf{h}$ onto $\mathbf{u}$.

To understand why this formula is true, let θ be the angle between $\mathbf{s}$ and $\mathbf{h}$. From triangle trigonometry we know that

$$|\mathbf{s}| \;=\; |\mathbf{h}|\cos\theta.$$

Since $\mathbf{u}$ has the same direction as $\mathbf{s}$, the angle between $\mathbf{u}$ and $\mathbf{h}$ is also θ, so

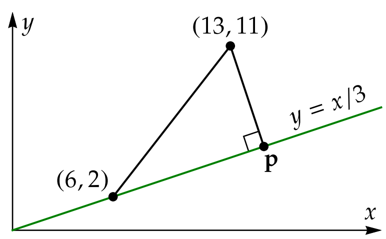
$$\mathbf{h}\cdot\mathbf{u} \;=\; |\mathbf{h}|\,|\mathbf{u}|\cos\theta.$$

But $\mathbf{u}$ is a unit vector, which means that $|\mathbf{u}| = 1$. Then

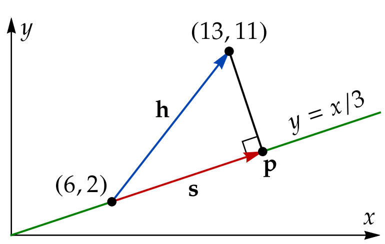
$$\mathbf{h}\cdot\mathbf{u} = |\mathbf{h}|\cos\theta,$$

and it follows that $|\mathbf{s}| = \mathbf{h}\cdot\mathbf{u}$.

The reason the projection formula is useful is that it allows us to determine the magnitude of $\mathbf{s}$ when we know the direction of $\mathbf{s}$ as well as the vector $\mathbf{h}$. That is, it lets us find one side of a right triangle when we know the hypotenuse. From this point of view, the projection formula is like a vector version of triangle trigonometry.



▲ **Figure 2:** The right triangle for Example 1.



▲ **Figure 3:** The vectors **h** and **s** for Example 1.

EXAMPLE 1

Figure 2 shows a right triangle in the plane. Find the coordinates of the point $\mathbf{p}$.

SOLUTION Let $\mathbf{h}$ and $\mathbf{s}$ be the vectors shown in Figure 3. Clearly

$$\mathbf{h} \;=\; (13,11)-(6,2) \;=\; (7,9).$$

Furthermore, the vector $\mathbf{s}$ lies along the line $y = x/3$. The vector $(3,1)$ also points in this direction, so a unit vector in the direction of $\mathbf{s}$ is

$$\mathbf{u} \;=\; \frac{1}{|(3,1)|}(3,1) \;=\; \frac{1}{\sqrt{10}}(3,1).$$

Then by the projection formula,

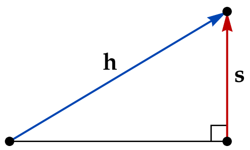
$$|\mathbf{s}| \;=\; \mathbf{h}\cdot\mathbf{u} \;=\; (7,9)\cdot\frac{1}{\sqrt{10}}(3,1) \;=\; \frac{30}{\sqrt{10}} \;=\; 3\sqrt{10}.$$

We conclude that

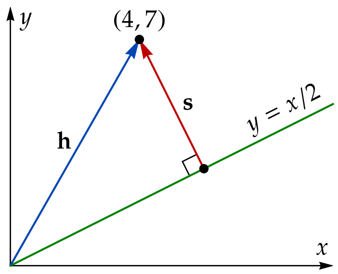
$$\mathbf{s} \;=\; 3\sqrt{10}\,\mathbf{u} \;=\; (9,3),$$

and therefore

$$\mathbf{p} \;=\; (6,2)+\mathbf{s} \;=\; (6,2)+(9,3) \;=\; \boxed{(15,5)}$$



▲ **Figure 4:** A right triangle with vectors $\mathbf{h}$ and $\mathbf{s}$.



▲ **Figure 5:** The vectors $\mathbf{h}$ and $\mathbf{s}$ for Example 2.

Incidentally, although we have stated the projection formula for a hypotenuse vector $\mathbf{h}$ and side vector $\mathbf{s}$ that emanate from the same vertex, it works just as well if $\mathbf{h}$ and $\mathbf{s}$ end at the same vertex, as shown in Figure 4. In particular, if $\mathbf{u}$ is a unit vector in the direction of $\mathbf{s}$, then

$$|\mathbf{s}| = \mathbf{h} \cdot \mathbf{u}.$$

EXAMPLE 2

Find the distance from the point $(4,7)$ to the line $y = x/2$.

SOLUTION Let $\mathbf{h} = (4,7)$ and $\mathbf{s}$ be the vectors shown in Figure 5. To find a unit vector in the direction of $\mathbf{s}$, note first that $(2,1)$ points in the direction of the line $y = x/2$. Turning $90°$ counterclockwise, we find that $(-1,2)$ is parallel to $\mathbf{s}$, and therefore

$$\mathbf{u} = \frac{1}{|(-1,2)|}(-1,2) = \frac{1}{\sqrt{5}}(-1,2)$$

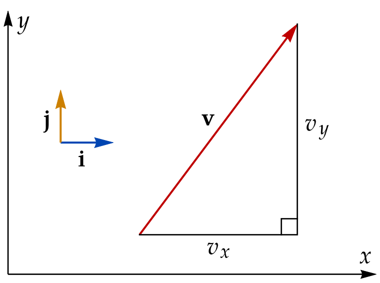
is a unit vector in the direction of $\mathbf{s}$. By the projection formula, it follows that

$$|\mathbf{s}| = \mathbf{h} \cdot \mathbf{u} = (4,7) \cdot \frac{1}{\sqrt{5}}(-1,2) = \frac{10}{\sqrt{5}} = \boxed{2\sqrt{5}}.$$

Change of Basis

The projection formula is related to an important idea known as **change of basis**. So far, the only basis for $\mathbb{R}^2$ we have been using is the **standard basis**, which consists of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the x and y directions. Every vector $\mathbf{v}$ can be expressed as a linear combination of the standard basis vectors:

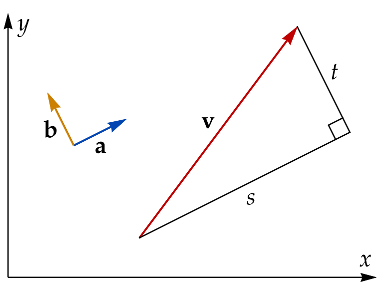
$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}.$$



▲ **Figure 6:** The components of $\mathbf{v}$ in the x and y directions.

The coefficients v_x and v_y of this linear combination are the components of $\mathbf{v}$ in the x and y directions, as shown in Figure 6.

Now, it is an important principle of geometry that there's nothing particularly special about the x and y directions. Indeed, if we choose another set of perpendicular directions, the corresponding pair of unit vectors $\mathbf{a}$, $\mathbf{b}$ should have the same properties as $\mathbf{i}$ and $\mathbf{j}$.



▲ **Figure 7:** The components of $\mathbf{v}$ in the directions of $\mathbf{a}$ and $\mathbf{b}$.

Orthonormal Bases and Components

A pair of vectors $\mathbf{a}, \mathbf{b}$ in the plane is called an **orthonormal basis** for $\mathbb{R}^2$ if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal unit vectors. In this case, every vector $\mathbf{v}$ in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{v} = s\,\mathbf{a} + t\,\mathbf{b}.$$

The coefficients s, t in this linear combination are the **components of v in the directions of a and b**, as shown in Figure 7.

It follows from the projection formula that s and t are given by dot products:

$$\boxed{s = \mathbf{v} \cdot \mathbf{a}} \qquad \text{and} \qquad \boxed{t = \mathbf{v} \cdot \mathbf{b}}$$

In the case where $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$, these formulas are just

$$v_x = \mathbf{v} \cdot \mathbf{i} \qquad \text{and} \qquad v_y = \mathbf{v} \cdot \mathbf{j}.$$

EXAMPLE 3

Find the components of the vector $(3,7)$ with respect to the orthonormal basis

$$\mathbf{a} = \frac{1}{\sqrt{2}}(1,1), \qquad \mathbf{b} = \frac{1}{\sqrt{2}}(-1,1).$$

Note that **a** and **b** are indeed orthogonal unit vectors.

SOLUTION We have

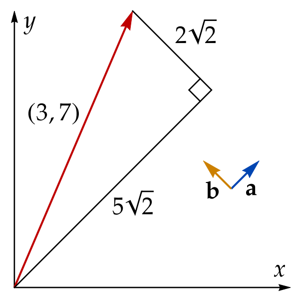
$$(3,7) \cdot \mathbf{a} = (3,7) \cdot \frac{1}{\sqrt{2}}(1,1) = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

and

$$(3,7) \cdot \mathbf{b} = (3,7) \cdot \frac{1}{\sqrt{2}}(-1,1) = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

Thus the components of $(3,7)$ are $\boxed{5\sqrt{2}}$ in the direction of **a** and $\boxed{2\sqrt{2}}$ in the direction of **b**, as shown in Figure 8. Indeed, it is easy to check that

$$(3,7) = 5\sqrt{2}\,\mathbf{a} + 2\sqrt{2}\,\mathbf{b}.$$



▲ **Figure 8:** The components of $(3,7)$ in the directions of **a** and **b**.

Orthonormal bases for $\mathbb{R}^3$ work in a similar way, except that an orthonormal basis consists of *three* orthogonal unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. For such a basis, every vector $\mathbf{v}$ in $\mathbb{R}^3$ can be written as a linear combination

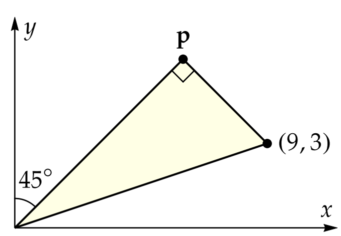
$$\mathbf{v} = s\,\mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c}$$

where s, t, and u are the components of $\mathbf{v}$ in the directions of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. These are given by the formulas

$$s = \mathbf{v} \cdot \mathbf{a}, \qquad t = \mathbf{v} \cdot \mathbf{b}, \qquad \text{and} \qquad u = \mathbf{v} \cdot \mathbf{c}.$$

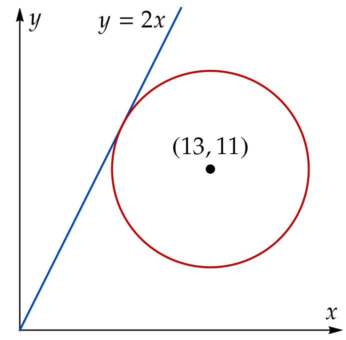
EXERCISES

1. The following figure shows a right triangle with one vertex at the origin.



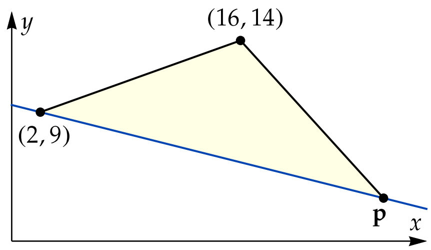
Find the coordinates of the point **p**.

2. In the following figure, a circle centered at the point $(13, 11)$ is tangent to the line $y = 2x$.



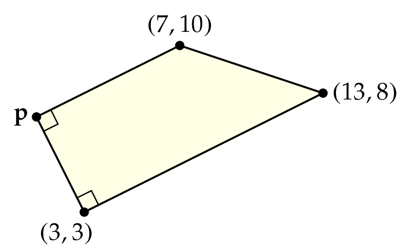
Find the area of the circle.

3. The following figure shows an isosceles triangle sitting on the line $x + 4y = 38$.



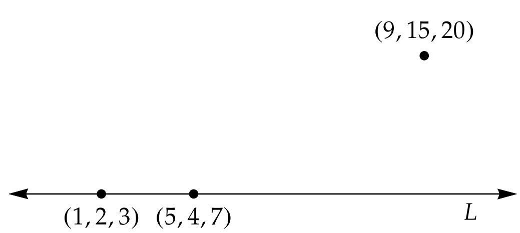
Find the coordinates of the point $\mathbf{p}$.

4. The following figure shows a trapezoid with two right angles.



Find the coordinates of the point $\mathbf{p}$.

5. The following figure shows a line L in $\mathbb{R}^3$.



Find the distance from L to the point $(9, 15, 20)$.

6–8 ■ Find the components of the vector **v** with respect to the given orthonormal basis.

6. $\mathbf{v} = (1, 8)$; $\mathbf{a} = \left(\frac{3}{5}, \frac{4}{5}\right)$, $\mathbf{b} = \left(-\frac{4}{5}, \frac{3}{5}\right)$

7. $\mathbf{v} = (9, -3)$; $\mathbf{a} = \frac{1}{\sqrt{2}}(1, 1)$, $\mathbf{b} = \frac{1}{\sqrt{2}}(-1, 1)$

8. $\mathbf{v} = (8, 2, 3)$; $\mathbf{a} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$, $\mathbf{b} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, $\mathbf{c} = \left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

7.2 Determinants

A **matrix** is simply a rectangular array of numbers. For example,

$$\begin{bmatrix} 1 & 7 & 3 & 8 \\ 4 & 1 & 5 & 3 \\ 2 & 9 & 0 & 1 \end{bmatrix}$$

is a matrix made up of 12 numbers, which are called the **entries** of the matrix. This particular matrix is a **3×4 matrix**, since the entries are organized into 3 rows and 4 columns.

Here "3×4" is pronounced "three-by-four".

A **square matrix** is a matrix with the same number of rows and columns. For example,

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 & 8 \\ 3 & 9 & 2 \\ 4 & 6 & 1 \end{bmatrix}$$

are square matrices, the first being a 2×2 matrix, and the second being a 3×3 matrix.

Matrices are important throughout mathematics and the sciences, and we will be using them quite a lot as we delve into linear algebra. For the moment though, the only aspect of matrices that we care about are determinants.

2×2 Determinants

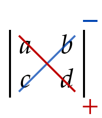
Every square matrix has an associated scalar, called the **determinant** of the matrix. The determinant of a 2×2 matrix is defined by the formula

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

Note that we use brackets [−] for a matrix, and vertical lines | − | for the determinant of a matrix.

For example,

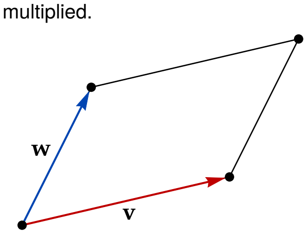
$$\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix} = (3)(7) - (2)(5) = 11.$$



▲ **Figure 1:** In a 2×2 determinant, the entries diagonally across from one another are multiplied.

Note that the entries of the matrix diagonally across from one another are multiplied, as shown in Figure 1.

There is a nice geometric interpretation of 2×2 determinants.



▲ **Figure 2:** A parallelogram with vectors **v** and **w** along its sides.

Area of a Parallelogram in $\mathbb{R}^2$

Let P be a parallelogram in the plane with the vectors **v** and **w** emanating from one vertex, as shown in Figure 2. Then

$$\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = \pm\text{area}(P)$$

The plus or minus on the right side of this formula is essential, since the value of a 2×2 determinant is sometimes negative. But the absolute value of the determinant is always equal to the area of a corresponding parallelogram.

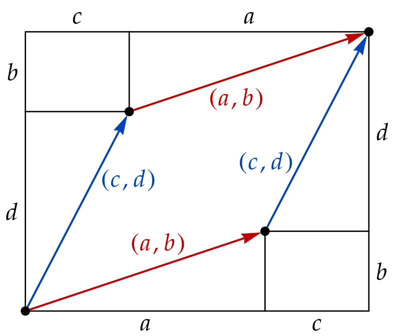


Figure 3: A rectangle subdivided into a parallelogram, four right triangles, and two small rectangles.

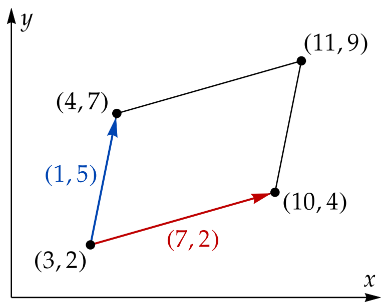


Figure 4: The parallelogram from Example 1.

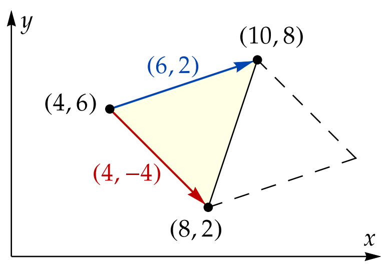


Figure 5: The triangle from Example 2.

Figure 3 shows a partial justification for this formula. In the figure, the large rectangle has area $(a+b)(c+d)$, so the area of the parallelogram is

$$(a+b)(c+d) - \frac{1}{2}ac - \frac{1}{2}ac - \frac{1}{2}bd - \frac{1}{2}bd - bc - bc = ad - bc.$$

This justification assumes that a, b, c, and d are all positive, but similar arguments can be made in cases where one or more of these components is negative.

EXAMPLE 1

Find the area of the parallelogram in $\mathbb{R}^2$ with vertices at $(3,2)$, $(10,4)$, $(11,9)$, and $(4,7)$.

SOLUTION This parallelogram is shown in Figure 4. The two highlighted vectors are

$$(10,4)-(3,2) = (7,2) \qquad \text{and} \qquad (4,7)-(3,2) = (1,5)$$

and

$$\begin{vmatrix} 7 & 2 \\ 1 & 5 \end{vmatrix} = (7)(5)-(1)(2) = 33,$$

so the area is $\boxed{33}$.

Since every triangle is half of a parallelogram, we can also use 2×2 determinants to find the areas of triangles in the plane.

EXAMPLE 2

Find the area of the triangle with vertices $(4,6)$, $(10,8)$, and $(8,2)$.

SOLUTION The given triangle is half of the area of a parallelogram, as shown in Figure 5. The two highlighted vectors are

$$(8,2)-(4,6) = (4,-4) \qquad \text{and} \qquad (10,8)-(4,6) = (6,2).$$

Since

$$\begin{vmatrix} 4 & -4 \\ 6 & 2 \end{vmatrix} = (4)(2)-(-4)(6) = 32,$$

the area of the parallelogram is 32, so the area of the triangle is half of 32, which is $\boxed{16}$.

The Sign of the Determinant

There is another geometric interpretation of the determinant that clarifies the circumstances in which a determinant is negative.

Determinant Sine Formula

If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$, then

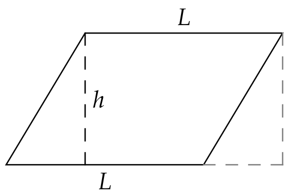
$$\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = |\mathbf{v}|\,|\mathbf{w}|\sin\theta,$$

where θ is the counterclockwise angle from $\mathbf{v}$ to $\mathbf{w}$.

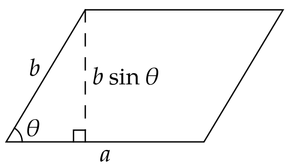
Here "counterclockwise angle from **v** to **w**" means the angle by which **v** must be rotated counterclockwise to point in the same direction as **w**. We usually think of this angle as being between $-180°$ and $180°$, with negative angles corresponding to clockwise rotation.

Note that this formula is similar to the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$ for dot products, except that it involves the sine instead of the cosine. Thus dot products and 2×2 determinants are sort of counterparts to one another.

To see the relation between the sine formula and the area of a parallelogram, recall first that a parallelogram with base L and height h has area Lh, as shown in Figure 6. Using a little triangle trigonometry, it follows that a parallelogram with side lengths a, b and angle θ has area $ab \sin\theta$, as shown in Figure 7.



▲ **Figure 6:** This parallelogram has area Lh, since the triangle on the left can be moved to the right to make a rectangle with base L and height h.

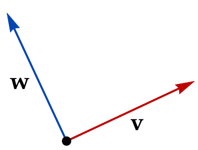


▲ **Figure 7:** This parallelogram has area $ab \sin\theta$.

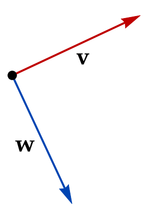
The advantage of the sine formula is that it allows us to predict the sign of the determinant. Specifically, since $\sin\theta$ is positive for $0° < \theta < 180°$ and negative for $-180° < \theta < 0°$, we find that:

- The determinant $\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$ is **positive** if **w** is **counterclockwise** from **v**, and
- The determinant $\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$ is **negative** if **w** is **clockwise** from **v**.

These two cases are illustrated in Figures 8 and 9. Note that the determinant is zero if **v** and **w** point in either the same direction or exact opposite directions.



▲ **Figure 8:** Two vectors **v** and **w** for which the corresponding determinant is positive.



▲ **Figure 9:** Two vectors **v** and **w** for which the corresponding determinant is negative.

3×3 Determinants

The determinant of a 3×3 matrix is defined by the formula

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh + bfg - bdi + cdh - ceg.$$

This formula is quite complicated, but it can be written more simply using 2×2 determinants:

$$\boxed{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

This formula is known as the **cofactor expansion** for a 3×3 determinant. It is easy to remember if you understand the following two rules:

Rule 1. There is one term for each entry on the first row of the matrix:

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ d & e & f \\ g & h & i \end{vmatrix} = \mathbf{a}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \mathbf{b}\begin{vmatrix} d & f \\ g & i \end{vmatrix} + \mathbf{c}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

These terms are alternately added and subtracted, with the first term added, the second term subtracted, and so forth.

Rule 2. Each of the smaller determinants is obtained by crossing out one row and one column from the larger matrix:

The smaller determinants are called **minors** of the large matrix.

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

In particular, each entry on the first row is multiplied by the determinant of the matrix obtained from crossing out the row and column that contain the entry.

EXAMPLE 3

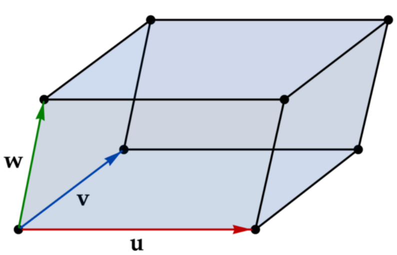
Compute $\begin{vmatrix} 3 & 7 & 6 \\ 8 & 5 & 4 \\ 0 & 2 & 1 \end{vmatrix}$.

SOLUTION Using cofactor expansion,

$$\begin{vmatrix} 3 & 7 & 6 \\ 8 & 5 & 4 \\ 0 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 5 & 4 \\ 2 & 1 \end{vmatrix} - 7\begin{vmatrix} 8 & 4 \\ 0 & 1 \end{vmatrix} + 6\begin{vmatrix} 8 & 5 \\ 0 & 2 \end{vmatrix}$$

$$= 3(-3) - 7(8) + 6(16) = \boxed{31}$$

Just as a 2×2 determinant can be viewed as the area of a parallelogram, a 3×3 determinant can be viewed as the volume of a three-dimensional shape known as a **parallelepiped** (par-ə-lel-ə-PIE-ped), as shown in Figure 10. This polyhedron resembles a cube or rectangular box, but it has three pairs of parallel faces, all of which are parallelograms.



▲ **Figure 10:** A parallelepiped.

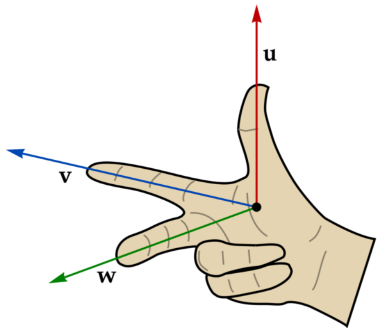
Volume of a Parallelepiped in $\mathbb{R}^3$

Let P be a parallelepiped in $\mathbb{R}^3$ with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ emanating from one vertex, as shown in Figure 10. Then

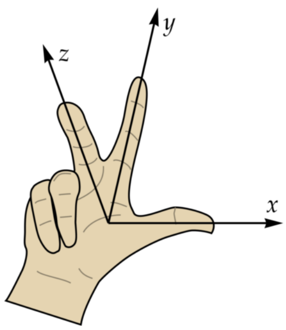
$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \pm\text{volume}(P).$$

As in the two dimensional case, the sign of the determinant depends on the orientation of the vectors. Specifically:

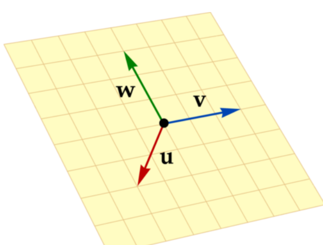
- If the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are a **right-handed triple**, then the corresponding determinant will be **positive**.



▲ **Figure 11:** Three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ that form a right-handed triple.



▲ **Figure 12:** The x, y, and z axes have a right-handed orientation.



▲ **Figure 13:** Three coplanar vectors.

- If the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are a **left-handed triple**, then the corresponding determinant will be **negative**.

Whether three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ form a right-handed triple can be checked using the **right-hand rule**, which is shown in Figure 11. To use the right-hand rule, point your thumb in the direction of the first vector $\mathbf{u}$ and your index finger in the direction of the second vector $\mathbf{v}$. If your middle finger now points in the general direction of the third vector $\mathbf{w}$, then the three vectors are a right-handed triple.

Note that the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a right-handed triple. This is because of the right-handed orientation of the x, y, and z axes. Specifically, if we look straight at the xy-plane with the x-axis pointing to the right and the y-axis pointing up, then the z-axis points straight towards us. This is "right-handed" in the sense that if we look at our right hand with our thumb pointing to the right and our index finger pointing up, then our middle finger points straight towards us, as shown in Figure 12.

Finally, observe that a 3×3 determinant can only be zero if the corresponding parallelepiped has zero volume. This occurs when the three vectors are **coplanar**, meaning that they all lie in the same plane in $\mathbb{R}^3$, as shown in Figure 13.

Determinants in General

Determinants of 4×4 and larger matrices can be defined using cofactor expansion, using the same rules as the cofactor expansion for a 3×3 matrix. For example, the cofactor expansion for a 4×4 determinant is

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

EXAMPLE 4

Compute $\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 5 & 1 & 2 & 4 \\ 3 & 4 & 6 & 5 \end{vmatrix}$.

SOLUTION With only two nonzero entries in the first row, the cofactor expansion for this determinant has only two nonzero terms:

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 5 & 1 & 2 & 4 \\ 3 & 4 & 6 & 5 \end{vmatrix} = -2\begin{vmatrix} 0 & 0 & 4 \\ 5 & 2 & 4 \\ 3 & 6 & 5 \end{vmatrix} + 3\begin{vmatrix} 0 & 0 & 4 \\ 5 & 1 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

For the same reason, both of these 3×3 determinants are fairly easy:

$$\begin{vmatrix} 0 & 0 & 4 \\ 5 & 2 & 4 \\ 3 & 6 & 5 \end{vmatrix} = 4\begin{vmatrix} 5 & 2 \\ 3 & 6 \end{vmatrix} = 96 \qquad \begin{vmatrix} 0 & 0 & 4 \\ 5 & 1 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 4\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix} = 68$$

Thus the determinant is $-2(96) + 3(68) = \boxed{12}$.

EXERCISES

1–2 ■ Compute the given 2×2 determinant.

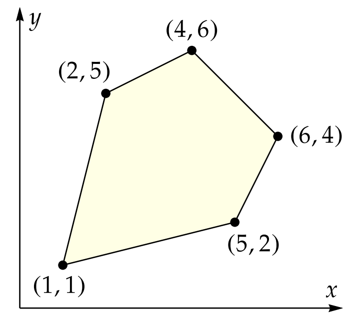
1. $\begin{vmatrix} -3 & 1 \\ 7 & -5 \end{vmatrix}$

2. $\begin{vmatrix} 4 & 2 \\ 8 & 3/2 \end{vmatrix}$

3. Find the area of the parallelogram with vertices $(0,1)$, $(3,4)$, $(-1,3)$, and $(2,6)$.

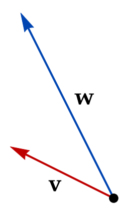
4. Find the area of the triangle with vertices $(0,3)$, $(2,0)$, and $(3,4)$.

5. Find the area of the following pentagon.

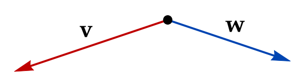


6–9 ■ Determine whether $\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}$ will be positive, negative, or zero for the given pair of vectors $\mathbf{v}, \mathbf{w}$.

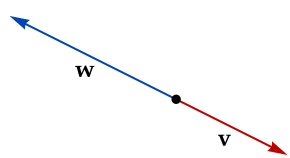
6.



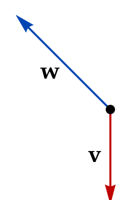
7.



8.



9.

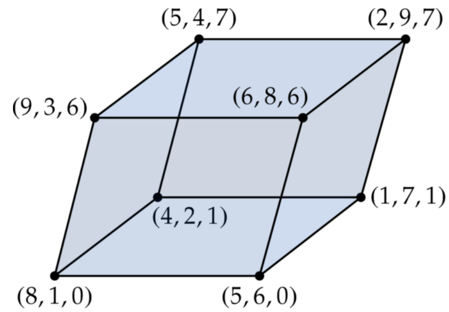


10–11 ■ Compute the given 3×3 determinant.

10. $\begin{vmatrix} 6 & 2 & 2 \\ 9 & 4 & -1 \\ 1 & 3 & 1 \end{vmatrix}$

11. $\begin{vmatrix} -2 & -3 & 7 \\ -3 & -4 & -3 \\ 3 & 1 & 9 \end{vmatrix}$

12. Find the volume of the following parallelepiped.



13–16 ■ Determine whether $\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$ will be positive, negative, or zero for vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in the given directions.

13. The vector **u** points directly to the left, the vector **v** points directly forwards, and the vector **w** points directly upwards.

14. The vector **u** points directly northwest, the vector **v** points directly upwards, and the vector **w** points directly to the east.

15. The vector **u** points directly to the left, the vector **v** points directly upwards, and the vector **w** points directly to the right.

16. The vector **u** points directly south, the vector **v** points directly east, and the vector **w** points directly upwards.

17–19 ■ Compute the given determinant.

17. $\begin{vmatrix} 0 & 3 & 0 & 0 \\ 0 & 6 & 0 & 2 \\ 3 & 0 & 4 & 7 \\ 5 & 2 & 8 & 0 \end{vmatrix}$

18. $\begin{vmatrix} 5 & 2 & 0 & 0 \\ -1 & 0 & 0 & -2 \\ 4 & -3 & 3 & -3 \\ -2 & 6 & 0 & 7 \end{vmatrix}$

19. $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$

7.3 Cross Products

In this section we introduce the cross product of vectors in $\mathbb{R}^3$. Like the dot product, the cross product can be thought of as a kind of multiplication of vectors, although it only works for vectors in three dimensions. As we shall see, this product is quite useful for three-dimensional geometry.

The Cross Product

The **cross product** of two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ is defined by the formula

$$\mathbf{v} \times \mathbf{w} \;=\; \left(v_y w_z - v_z w_y,\, v_z w_x - v_x w_z,\, v_x w_y - v_y w_x\right)$$

Note that each of the three components of the cross product is actually a 2×2 determinant.

$$\mathbf{v} \times \mathbf{w} \;=\; \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_x & v_z \\ w_x & w_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} \mathbf{k}$$

Writing the formula this way makes it look quite similar to the cofactor expansion of a 3×3 determinant. Indeed, we can write the above formula as follows.

$$\boxed{\mathbf{v} \times \mathbf{w} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}}$$

Note that this 3×3 determinant has vectors in the first row, which makes it quite different from other determinants we have seen. But if we ignore this distinction, evaluating this determinant using cofactor expansion yields the correct cross product.

EXAMPLE 1

Compute the cross product $(3, 1, 4) \times (2, 2, 7)$.

SOLUTION The cross product is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 2 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} \mathbf{k} = \boxed{(-1, -13, 4)}$$

When computing cross products, it is common to do them "in place" instead of writing the whole thing out. That is, we skip straight from writing the determinant to writing the answer:

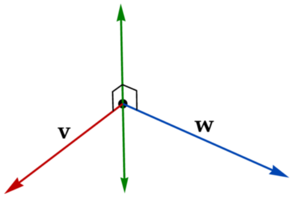
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 2 & 2 & 7 \end{vmatrix} = (-1, -13, 4)$$

Here each of the components on the right comes from a 2×2 determinant:

$$\begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} = -1, \qquad -\begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix} = -13, \qquad \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 4.$$

Geometric Interpretation

The most important geometric property of the cross product is the following.



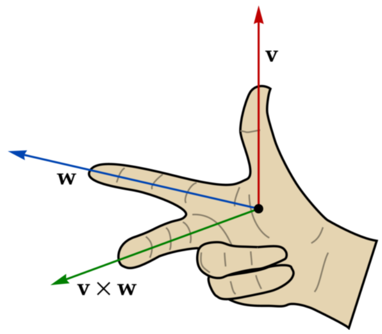
▲ **Figure 1:** The two directions orthogonal to both $\mathbf{v}$ and $\mathbf{w}$

Orthogonality of the Cross Product

The cross product $\mathbf{v}\times\mathbf{w}$ is orthogonal to both of the vectors $\mathbf{v}$ and $\mathbf{w}$.

This orthogonality is easy to verify by taking the dot products of $\mathbf{v}$ and $\mathbf{w}$ with $\mathbf{v}\times\mathbf{w}$. For example,

$$\begin{aligned}
\mathbf{v}\cdot(\mathbf{v}\times\mathbf{w}) &= (v_x, v_y, v_z)\cdot(v_y w_z - v_z w_y,\ v_z w_x - v_x w_z,\ v_x w_y - v_y w_x) \\
&= v_x(v_y w_z - v_z w_y) + v_y(v_z w_x - v_x w_z) + v_z(v_x w_y - v_y w_x) \\
&= {\color{red}v_x v_y w_z} - {\color{blue}v_x v_z w_y} + {\color{green}v_y v_z w_x} - {\color{red}v_y v_x w_z} + {\color{blue}v_z v_x w_y} - {\color{green}v_z v_y w_x} \\
&= 0.
\end{aligned}$$

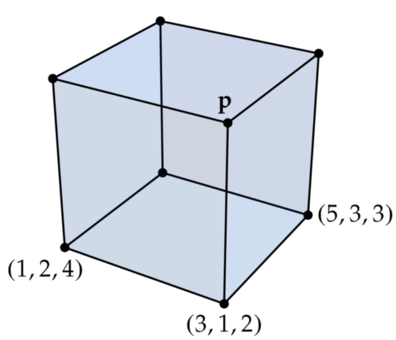


▲ **Figure 2:** The right-hand rule for finding the direction of a cross product.

Here each pair of terms of the same color canceled, leaving zero, and therefore $\mathbf{v}\times\mathbf{w}$ is orthogonal to $\mathbf{v}$. A similar computation shows that $\mathbf{v}\times\mathbf{w}$ is orthogonal to $\mathbf{w}$.

Of course, knowing that $\mathbf{v}\times\mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ does not determine its direction completely. There are typically two possible directions orthogonal to a given pair of vectors $\mathbf{v}$, $\mathbf{w}$, as shown in Figure 1. For example, if we know that a vector is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$, it is possible that it points in the direction of $\mathbf{k}$, but it is also possible that it points in the direction of $-\mathbf{k}$.

The resolution of this ambiguity is simple: it turns out that $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v}\times\mathbf{w}$ always form a right-handed triple. That is, if you orient your right hand so that your thumb points in the direction of $\mathbf{v}$ and your index finger points in the direction of $\mathbf{w}$, then your middle finger will point in the direction of $\mathbf{v}\times\mathbf{w}$, as shown in Figure 2. This technique is known as the **right-hand rule** for computing the direction of a cross product.



▲ **Figure 3:** The cube for Example 2.

EXAMPLE 2

Figure 3 shows a cube in $\mathbb{R}^3$. Find the coordinates of the point $\mathbf{p}$.

SOLUTION Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the three vectors shown in Figure 4. Clearly

$$\mathbf{u} = (5,3,3)-(3,1,2) = (2,2,1) \qquad\text{and}\qquad \mathbf{v} = (1,2,4)-(3,1,2) = (-2,1,2).$$

Now $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and by the right-hand rule $\mathbf{u}\times\mathbf{v}$ should be in the same direction as $\mathbf{w}$. We have

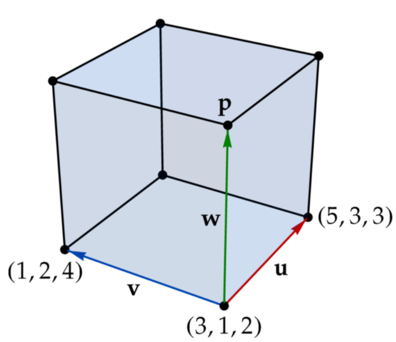
$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{vmatrix} = (3,-6,6),$$

so $\mathbf{w}$ is parallel to $(3,-6,6)$.

Now, the length of $\mathbf{w}$ should be the same as the lengths of $\mathbf{u}$ and $\mathbf{v}$, which is 3. Since $|(3,-6,6)| = 9$, we must divide this vector by 3 to obtain $\mathbf{w}$. We conclude that

$$\mathbf{w} = \frac{1}{3}(3,-6,6) = (1,-2,2).$$

Note that this is indeed orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{p} = (3,1,2)+\mathbf{w} = \boxed{(4,-1,4)}$.



▲ **Figure 4:** The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Incidentally, the fact that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ can be quite useful for checking cross product calculations. For example, suppose we compute

$$(3,1,2) \times (-1,1,4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ -1 & 1 & 4 \end{vmatrix} = (2,-14,4).$$

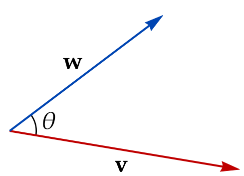
A quick way to check that our computation is correct is to take the dot product of our result with each of the original vectors:

$$(2,-14,4)\cdot(3,1,2) = 0 \qquad \text{and} \qquad (2,-14,4)\cdot(-1,1,4) = 0.$$

Since our result is orthogonal to both of the original vectors, we have almost certainly computed the cross product correctly. This simple check usually catches errors in a cross product calculation, and is worth doing almost every time you take a cross product.

Magnitude of the Cross Product

There is a simple formula for the magnitude of the cross product.



▲ **Figure 5:** The angle between $\mathbf{v}$ and $\mathbf{w}$.

Magnitude of the Cross Product

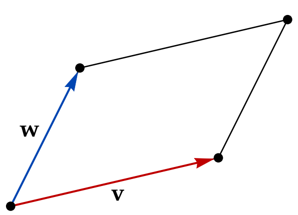
If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, then

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin\theta$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, as shown in Figure 5

This is similar to our formula for a 2×2 determinant, except that the angle θ here is always between 0° and 180°. For vectors in $\mathbb{R}^3$, there is no way to tell whether $\mathbf{w}$ is clockwise or counterclockwise from $\mathbf{v}$, since a rotation that looks clockwise from one direction will appear counterclockwise from the other direction.

As with 2×2 determinants, this formula means that the magnitude of a cross product can be interpreted as the area of a parallelogram.



▲ **Figure 6:** A parallelogram with vectors $\mathbf{v}$ and $\mathbf{w}$ along its sides.

Area of a Parallelogram in $\mathbb{R}^3$.

Let P be a parallelogram in $\mathbb{R}^3$ with vectors $\mathbf{v}$ and $\mathbf{w}$ emanating from one vertex, as shown in Figure 6. Then

$$\text{area}(P) = |\mathbf{v} \times \mathbf{w}|.$$

EXAMPLE 3

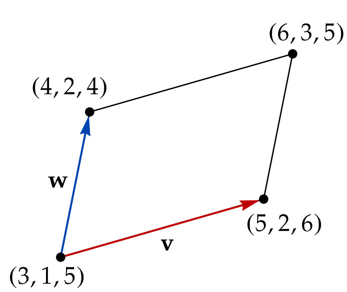
Find the area of the parallelogram in $\mathbb{R}^3$ with vertices at $(3,1,5)$, $(5,2,6)$, $(4,2,4)$, and $(6,3,5)$.

SOLUTION Since we aren't given a picture of the parallelogram, we must first determine how the four vertices are arranged. Since

$$(5,2,6)-(3,1,5) = (2,1,1) \qquad \text{and} \qquad (6,3,5)-(4,2,4) = (2,1,1)$$

these must represent parallel sides, so the vertices are arranged as in Figure 7. Then

$$\mathbf{v} = (5,2,6)-(3,1,5) = (2,1,1) \qquad \text{and} \qquad \mathbf{w} = (4,2,4)-(3,1,5) = (1,1,-1),$$



▲ **Figure 7:** The parallelogram from Example 3.

so

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (-2, 3, 1).$$

Note that this is indeed orthogonal to both $(2,1,1)$ and $(1,1,-1)$. Then the area of the parallelogram is

$$|\mathbf{v} \times \mathbf{w}| = |(-2,3,1)| = \sqrt{(-2)^2 + (3)^2 + (1)^2} = \boxed{\sqrt{14}}$$

When computing a cross product, it is always a good idea to check that the final result is orthogonal to the two given vectors.

Algebraic Properties

Like dot product, the cross product has several properties in common with multiplication of numbers. However, it also has a few more unusual properties.

Algebraic Properties of the Cross Product

1. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$ in $\mathbb{R}^3$.
2. $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$ for any two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$.
3. $(k\mathbf{v}) \times \mathbf{w} = k(\mathbf{v} \times \mathbf{w})$ for any scalar k and any vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$.
4. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ for any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$.

It is also true that

$$\mathbf{v} \times (k\mathbf{w}) = k(\mathbf{v} \times \mathbf{w})$$

for any scalar k and vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$, and

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$.

The first property is quite striking, and really nothing like multiplication: the cross product of a vector with itself gives the zero vector! This is easy to check from the definition, since

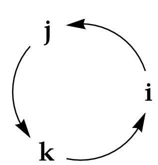
$$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} v_y & v_z \\ v_y & v_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_x & v_z \\ v_x & v_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_x & v_y \\ v_x & v_y \end{vmatrix} \mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.$$

More generally, $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ whenever $\mathbf{v}$ and $\mathbf{w}$ point in either the same direction or opposite directions.

The second property is like the opposite of the usual commutative law, and for that reason is known as the **anticommutative law**. Geometrically it corresponds to the fact that switching the positions of your thumb and index finger in the right-hand rule will switch the direction that your middle finger points.

By the way, here are the cross products of the standard basis vectors:

$\mathbf{i} \times \mathbf{i} = \mathbf{0}$	$\mathbf{j} \times \mathbf{j} = \mathbf{0}$	$\mathbf{k} \times \mathbf{k} = \mathbf{0}$
$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$



▲ **Figure 8:** The cyclic symmetry of **i**, **j**, and **k**.

Note the cyclic symmetry of the three standard basis vectors with respect to cross product. Figure 8 illustrates this similarity by placing the three vectors in a circle. Taking the cross product of any two consecutive vectors in this circle yields the third vector, but taking the cross product of two vectors in the wrong (i.e. clockwise) order yields the negative of the third vector.

A Closer Look — The Triple Product

There is a nice relationship between dot products, cross products, and 3×3 determinants. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \;=\; \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Because of this formula, the 3×3 determinant is sometimes referred to as the **triple product** of vectors.

This is known as the **triple product formula**. It arises because of the similarity between the definitions of the cross product and the 3×3 determinant:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \;=\; (u_x, u_y, u_z) \cdot \left(\begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_x & v_z \\ w_x & w_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} \mathbf{k} \right)$$

$$= u_x \begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix} - u_y \begin{vmatrix} v_x & v_z \\ w_x & w_z \end{vmatrix} + u_z \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} \;=\; \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

The triple product formula gives a simpler explanation for why $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. According to the formula,

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) \;=\; \begin{vmatrix} v_x & v_y & v_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \text{and} \qquad \mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) \;=\; \begin{vmatrix} w_x & w_y & w_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Both of these determinants have two rows that are the same, which means that the corresponding parallelepipeds have zero volume. This means that both determinants are zero, so $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

Incidentally, since the cross product of two vectors is a vector, it also makes sense to ask whether the cross product is associative. The answer to this question is **no**. For example, it is easy to check that

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} \;\neq\; \mathbf{i} \times (\mathbf{i} \times \mathbf{j}).$$

The product on the left is zero, but the product on the right comes out to $-\mathbf{j}$. Because the cross product isn't associative, it doesn't make sense to write a triple cross product like $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ without including parentheses around either $\mathbf{u} \times \mathbf{v}$ or $\mathbf{v} \times \mathbf{w}$.

EXERCISES

1–2 ■ Compute the given cross product.

1. $(1, 4, 2) \times (2, 1, 3)$

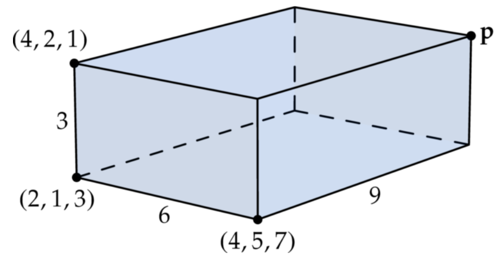
2. $(-6, -2, 1) \times (5, 1, -3)$

3–4 ■ Describe the direction of $\mathbf{v} \times \mathbf{w}$ from the given descriptions of $\mathbf{v}$ and $\mathbf{w}$.

3. The vector $\mathbf{v}$ points directly forwards, and the vector $\mathbf{w}$ points directly to the left.

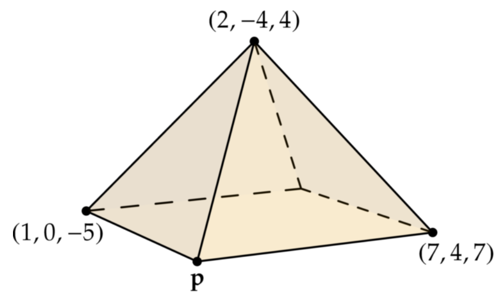
4. The vector $\mathbf{v}$ points directly southeast, the vector $\mathbf{w}$ points directly downwards.

5. The following figure shows a rectangular box in $\mathbb{R}^3$.



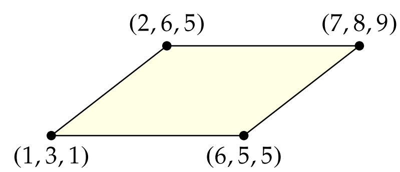
Find the coordinates of the point $\mathbf{p}$.

6. The following figure shows a right pyramid in $\mathbb{R}^3$ with a square base.

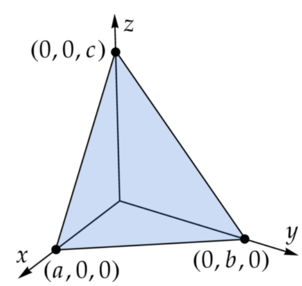


Find the coordinates of the point $\mathbf{p}$.

7. Find the area of the following parallelogram in $\mathbb{R}^3$.



8. The following figure shows a triangle in $\mathbb{R}^3$.



Find a formula for the area of the triangle in terms of a, b, and c. (Your final answer should not involve any vectors.)

9. Given that $|\mathbf{v}| = 4$, $|\mathbf{w}| = 3$, and $|\mathbf{v} \times \mathbf{w}| = 12$, what is the angle between $\mathbf{v}$ and $\mathbf{w}$?

10. Given that $|\mathbf{v}| = 5$, $|\mathbf{w}| = 6$, and $\mathbf{v} \cdot \mathbf{w} = 0$, what is the value of $|\mathbf{v} \times \mathbf{w}|$?

11. Given that $\mathbf{v} \times \mathbf{w} = (-2, 1, 4)$, compute $(\mathbf{v} + 3\mathbf{w}) \times (2\mathbf{v} + 4\mathbf{w})$.