8.1 Higher Dimensions

It is time for us to tackle the idea of n-dimensional space a little more directly. Here **n-dimensional space** refers to a geometric space $\mathbb{R}^n$ with n spatial dimensions, where n can be any positive integer. For example, $\mathbb{R}^1$ is an infinite line, $\mathbb{R}^2$ is an infinite plane, and $\mathbb{R}^3$ is a three-dimensional space that is infinite in all directions. When $n \geq 4$, the space $\mathbb{R}^n$ is said to be **higher-dimensional**.

Before we discuss the mathematics of higher-dimensional spaces, a few words about philosophy are in order. There is a basic philosophical objection to higher-dimensional spaces, which is that there are only three dimensions in the physical world. What does it even mean to discuss the geometry of four or five-dimensional space if these spaces don't really exist?

The answer is that we don't need these spaces to exist physically to be able to talk about them. Four and five-dimensional spaces exist on the same level as other mathematical objects, such as the number 10, the function $f(x) = x^2$, or the interval $[-1, 1]$. None of these things have any real physical existence—they are abstractions, which exist in the sense that they refer to certain aspects of real things. Thus we can have ten books, or the temperature can be ten degrees, but the number 10 itself isn't real in any physical sense.

The **philosophy of mathematics** is the branch of philosophy that considers the reality of mathematical objects and the nature of mathematical truth.

What four-dimensional space refers to is the set of possibilities for a system that can be described by four real variables. For example, if a chemical reaction involves four different reactants, then the concentrations (C_1, C_2, C_3, C_4) of the reactants are an ordered quadruple of real numbers. If a sector of the economy involves four goods, then the prices (p_1, p_2, p_3, p_4) of the goods are an ordered quadruple of real numbers. In each case, the set of all possible values for this quadruple can be thought of as a four-dimensional space, with each specific quadruple being a point in this space.

The reason we refer to $\mathbb{R}^n$ as a "space" is that we would like to extend our geometric intuition for $\mathbb{R}^2$ and $\mathbb{R}^3$ to higher dimensions as much as possible. It turns out that $\mathbb{R}^n$ is similar enough to $\mathbb{R}^2$ and $\mathbb{R}^3$ that it helps to think about it in geometric terms. But when we refer to a quadruple such as $(5, 3, 2, 7)$ as a "point" in $\mathbb{R}^4$, we are really just making an analogy to points in $\mathbb{R}^2$ and $\mathbb{R}^3$. Because higher-dimensional spaces only exist in the abstract, we must always be very careful to define geometric terms precisely before using them in this context. For example, the term "distance" seems self-explanatory in two and three dimensions, but for higher dimensions we must say exactly what we mean by "distance" before we can use this concept.

As with most statements in philosophy, the assertions being made here are hardly noncontroversial. For example, a **mathematical platonist**, who believes in the independent reality of mathematical objects, would reject the notion that $\mathbb{R}^4$ is any less of a geometric space than the world that we live in.

Thus, our description of higher dimensions will include precise definitions of many basic geometric concepts. In most cases, these definitions will be based on the descriptions of these concepts that we obtained in $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, the Pythagorean theorem is a theorem of Euclidean plane geometry, but in higher dimensions it becomes part of the definition of distance.

Points and Coordinates in $\mathbb{R}^n$

A **point** in n dimensions is simply a list of n real numbers. For example, $(5, 3, 2, 7)$ is a point in four dimensions, and $(8, -1, 3, 0, 9, 1/2)$ is a point in six dimensions. The individual numbers are called the **coordinates** of the point. The set of all points with n coordinates is denoted $\mathbb{R}^n$, and is referred to as **n-dimensional Euclidean space** or simply **n-dimensional space**.

In $\mathbb{R}^3$ the three coordinates of a point are usually called x, y, and z. When working with four or more dimensions, though, it is too cumbersome to use a different letter for each coordinate. Instead, we refer to the n coordinates in $\mathbb{R}^n$ as x_1, x_2, and so forth, with the last coordinate being x_n. For example, the point

$$(5, 3, 2, 7)$$

has 5 as its x_1-coordinate, 3 as its x_2-coordinate, 2 as its x_3-coordinate, and 7 as its x_4-coordinate.

Vectors in $\mathbb{R}^n$

A **vector** in $\mathbb{R}^n$ is a column of n real numbers:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

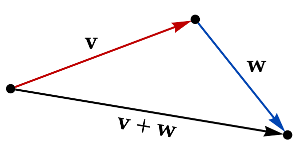
The numbers $v_1, v_2, \ldots, v_n$ are called the **components** of the vector. As in $\mathbb{R}^2$ and $\mathbb{R}^3$, it is convenient to think of vectors and points in $\mathbb{R}^n$ as being the same thing:

Here and elsewhere, we use ellipses $(\cdots)$ to represent components of a a vector $\mathbf{v}$ numbered between v_2 and v_n.

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, we defined addition of vectors geometrically using arrows, and then worked out that it corresponds to the componentwise sum. For higher dimensions, though, we use componentwise sum as the definition of addition.

$$\boxed{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}}$$



▲ **Figure 1:** The sum of two vectors in $\mathbb{R}^n$ can be thought of geometrically using arrows.

We imagine vector addition as having the same geometric meaning in higher dimensions that it does in $\mathbb{R}^2$ and $\mathbb{R}^3$, as shown in Figure 1.

Scalar multiplication is also defined componentwise:

$$\boxed{k\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}}$$

Again, we imagine this as having the same geometric meaning that it does in $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, multiplying a vector by 3 should increase its length by a factor of 3 without changing this direction, and multiplying a vector by -2 should double its length and make it point in the opposite direction.

There are n different **standard basis vectors** in $\mathbb{R}^n$, which we denote $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

$$\boxed{\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}$$

Any vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

Magnitude, Distances, and Angles

The **magnitude** of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is defined by the formula

$$\boxed{|\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}}$$

If we imagine vectors as arrows in $\mathbb{R}^n$, then the magnitude can be thought of as the length of the arrow.

EXAMPLE 1

Find the magnitude of the vector $(2, 4, 2, 5)$.

SOLUTION We have

$$|(2,4,2,5)| = \sqrt{2^2 + 4^2 + 2^2 + 5^2} = \sqrt{49} = \boxed{7}$$

If $\mathbf{p}$ and $\mathbf{q}$ are points in $\mathbb{R}^n$, the **distance** from $\mathbf{p}$ to $\mathbf{q}$ is defined by the formula

$$\boxed{\text{distance}(\mathbf{p},\mathbf{q}) = |\mathbf{p}-\mathbf{q}| = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2 + \cdots + (p_n-q_n)^2}}$$

Note then that the magnitude $|\mathbf{p}|$ of a point $\mathbf{p}$ represents its distance from the origin.

EXAMPLE 2

Find the distance between the points $(8, 7, 0, 3)$ and $(3, 1, 4, 1)$ in $\mathbb{R}^4$.

SOLUTION The distance is

$$|(8,7,0,3) - (3,1,4,1)| = |(5,6,-4,2)| = \sqrt{5^2 + 6^2 + (-4)^2 + 2^2} = \boxed{9}$$

The **dot product** of two vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ is defined by the formula

$$\boxed{\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n}$$

We can use dot product to define the angle between vectors. If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^n$, the **angle between v and w** is defined to be the value of θ between $0°$ and $180°$ that satisfies the equation

$$\boxed{\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos\theta}$$

We say that $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** if $\mathbf{v}\cdot\mathbf{w} = 0$. Two nonzero vectors are orthogonal if and only if the angle between them is $90°$.

A Closer Look **Cross Product in $\mathbb{R}^n$**

Though cross product as such only makes sense in $\mathbb{R}^3$, there is a nice generalization of cross product to any number of dimensions. This operation takes $n-1$ vectors in $\mathbb{R}^n$ as input, and outputs a new vector that is orthogonal to all of them. For example, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^4$, then the determinant

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}$$

A similar formula works in $\mathbb{R}^n$, with the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ on the first row of an $n \times n$ determinant, and the $n-1$ input vectors on the remaining rows.

yields a vector $\mathbf{c}$ in $\mathbb{R}^4$ that is orthogonal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Moreover, the magnitude of $\mathbf{c}$ is precisely the volume of the (3-dimensional) parallelepiped in $\mathbb{R}^4$ determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Interestingly, in the case of $\mathbb{R}^2$ this generalized cross product takes only a single vector $\mathbf{v}$ as input, and is given by the formula

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} \\ v_x & v_y \end{vmatrix} = (v_y, -v_x).$$

The resulting vector has the same magnitude as $\mathbf{v}$, but is turned 90° clockwise. Thus the operation of turning a vector 90° can be thought of as a two-dimensional analog of cross product!

EXAMPLE 3

Find the angle between the vectors $(2,3,4,5)$ and $(3,1,2,2)$ in $\mathbb{R}^4$.

SOLUTION Let $\mathbf{v} = (2,3,4,5)$ and $\mathbf{w} = (3,1,2,2)$. Then the equation

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$$

becomes

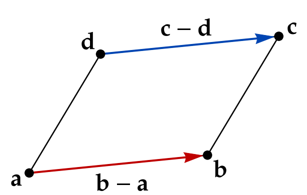
$$27 = \sqrt{54}\,\sqrt{18}\cos\theta.$$

Solving for $\cos\theta$ and simplifying yields $\cos\theta = \sqrt{3}/2$, and therefore $\theta = \boxed{30°}$.

Geometry in $\mathbb{R}^n$

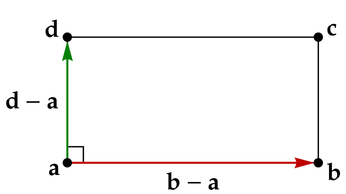
Essentially all of the geometry that we know in $\mathbb{R}^2$ and $\mathbb{R}^3$ continues to work in $\mathbb{R}^n$, assuming we interpret all of the geometric terms correctly using vectors. For example, four points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in $\mathbb{R}^n$ are said to be the **vertices of a parallelogram** if

$$\mathbf{c} - \mathbf{d} = \mathbf{b} - \mathbf{a},$$



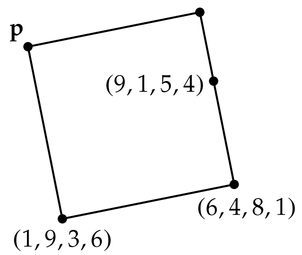
▲ **Figure 2:** Four points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the vertices of a parallelogram in $\mathbf{d} - \mathbf{c} = \mathbf{b} - \mathbf{a}$.

as shown in Figure 2. Such a parallelogram is called a **rectangle** if $\mathbf{d} - \mathbf{a}$ is orthogonal to $\mathbf{b} - \mathbf{a}$, as shown in Figure 3, and a rectangle is called a **square** if $|\mathbf{d} - \mathbf{a}| = |\mathbf{b} - \mathbf{a}|$.

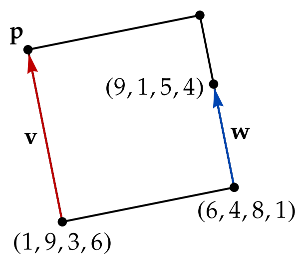


▲ **Figure 3:** A parallelogram with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ is a rectangle if $\mathbf{c} - \mathbf{a}$ is orthogonal to $\mathbf{b} - \mathbf{a}$.

The reader should tentatively assume that all geometric concepts from two and three dimensions continue to make sense in higher dimensions, and that these concepts interact in all of the familiar ways. For example, it makes perfect sense to talk about a circle in $\mathbb{R}^n$, and the area of such a circle is still πr^2, where r is the radius. Lines and planes make also make sense in $\mathbb{R}^n$, and so forth. Essentially all of the vector geometry we have learned in $\mathbb{R}^2$ and $\mathbb{R}^3$ continues to work for these same shapes in $\mathbb{R}^n$.



▲ **Figure 4:** The square from Example 4.



▲ **Figure 5:** The vectors **v** and **w**.

EXAMPLE 4

Figure 4 shows a square in $\mathbb{R}^4$. Find the coordinates of the point $\mathbf{p}$.

SOLUTION Let $\mathbf{v}$ and $\mathbf{w}$ be the parallel vectors shown in Figure 5. We can find $\mathbf{w}$ by subtracting the endpoints:

$$\mathbf{w} = (9,1,5,4) - (6,4,8,1) = (3,-3,-3,3).$$

This gives us the direction of $\mathbf{v}$. The magnitude of $\mathbf{v}$ is the side length of the square, which is the distance between the two bottom points:

$$\left|(6,4,8,1) - (1,9,3,6)\right| = \left|(5,-5,5,-5)\right| = \sqrt{5^2 + (-5)^2 + 5^2 + (-5)^2} = 10.$$

Thus $\mathbf{v}$ is parallel to $\mathbf{w}$ but has a magnitude of 10. Since

$$|\mathbf{w}| = \sqrt{3^2 + (-3)^2 + 3^2 + (-3)^2} = 6,$$

we conclude that $\mathbf{v} = (10/6)\mathbf{w} = (5/3)\mathbf{w} = (5,-5,-5,5)$. Then

$$\mathbf{p} = (1,9,3,6) + \mathbf{v} = (1,9,3,6) + (5,-5,-5,5) = \boxed{(6,4,-2,11)}$$

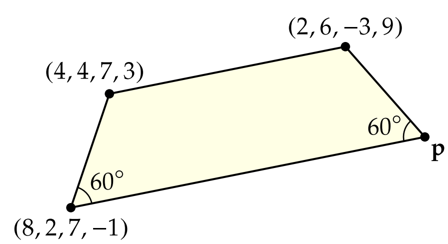
Of course, not all of the geometry in higher dimensions is so mundane. In addition to two and three dimensional shapes, there are also lots of interesting high-dimensional shapes in $\mathbb{R}^n$, including **hyperspheres** (higher-dimensional analogs of circles and spheres) and **hypercubes** (higher-dimensional analogs of squares and cubes), but we must learn quite a bit of vector geometry and vector calculus in $\mathbb{R}^n$ before we can investigate the properties of such shapes.

EXERCISES

1. Find the distance between the points $(5,1,3,7,6)$ and $(4,2,6,9,5)$ in $\mathbb{R}^5$.

2. Find the angle between the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -1 \\ 5 \\ 3 \end{bmatrix}$ in $\mathbb{R}^4$.

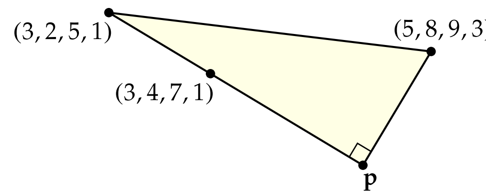
3. The following figure shows a trapezoid in $\mathbb{R}^4$.



Find the coordinates of the point $\mathbf{p}$.

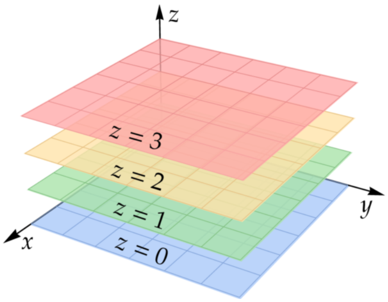
4. Find the area of the triangle in $\mathbb{R}^4$ with vertices $(1,1,0,0)$, $(0,1,1,0)$, and $(0,0,1,1)$.

5. The following figure shows a right triangle in $\mathbb{R}^4$.

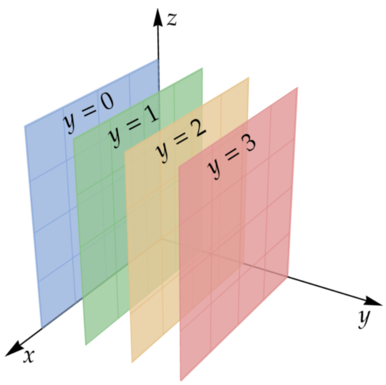


Find the coordinates of the point $\mathbf{p}$.

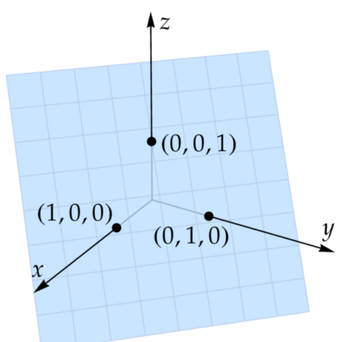
8.2 Planes and Hyperplanes



▲ **Figure 1:** The xy-plane and several other horizontal planes.



▲ **Figure 2:** The xz-plane and several parallel planes.



▲ **Figure 3:** The plane $x + y + z = 1$.

A **linear equation** in three variables x, y, and z is any equation of the form

$$ax + by + cz = d,$$

where a, b, c, d are constants and the coefficients a, b, c are not all zero. Any such equation defines a plane in $\mathbb{R}^3$.

Here are some examples of linear equations and the corresponding planes:

- The equation $z = 0$ defines the xy-plane in $\mathbb{R}^3$, since the points on the xy-plane are precisely those points whose z-coordinate is zero.
- If d is any constant, the equation $z = d$ defines a horizontal plane in $\mathbb{R}^3$, which is parallel to the xy-plane. Figure 1 shows several such planes.
- The equations $x = 0$ and $y = 0$ define the yz-plane and xz-plane, respectively, and equations of the form $x = d$ or $y = d$ define planes parallel to these. For example, Figure 2 shows several planes of the form $y = d$.
- The equation $x + y + z = 1$ defines a slanted plane in $\mathbb{R}^3$, which goes through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. This plane is shown in Figure 3.

In general, two planes that do not intersect are said to be **parallel**. Such planes can be defined by equations having the same coefficients of x, y, and z, but different constant terms, i.e.

$$ax + by + cz = d \qquad \text{and} \qquad ax + by + cz = e$$

for $d \neq e$. No point (x, y, z) can simultaneously satisfy both of these equations, so two planes of this form do not intersect.

EXAMPLE 1

Find an equation for the plane that is parallel to the plane $4x + y + 2z = 8$ and goes through the point $(3, 1, 2)$.

SOLUTION The desired plane must have an equation of the form

$$4x + y + 2z = d$$

for some constant d. Plugging in the point $(3, 1, 2)$ gives

$$4(3) + (1) + 2(2) = d,$$

so $d = 17$. Thus the desired plane is defined by

$$4x + y + 2z = 17.$$

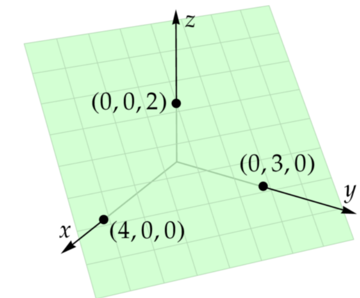
Note that the equation for a plane is not unique. For example, the planes defined by the equations

$$3x + 4y + 6z = 12 \qquad \text{and} \qquad 6x + 8y + 12z = 24.$$

are the same, since the second equation is just twice the first equation. In general, any nonzero scalar multiple of the equation for a plane gives another equation for the same plane.

Intercepts

The **intercepts** of a plane are the locations at which the plane intersects the x, y, and z axes. Most planes intersect each axis at exactly one point, and finding these intercepts can help to give a sense of how a plane sits in space.



▲ **Figure 4:** The plane $3x + 4y + 6z = 12$.

EXAMPLE 2

Find the points at which the plane $3x + 4y + 6z = 12$ intersects the three axes.

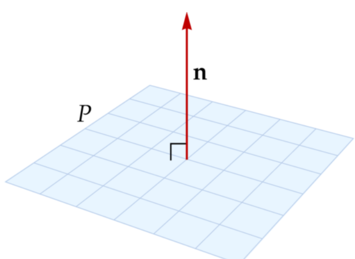
SOLUTION A point lies on the x-axis if and only if its y and z coordinates are both zero. Thus, we can figure out where the plane intersects the x-axis by setting y and z equal to 0 and then solving for x:

$$3x + 4(0) + 6(0) = 12.$$

Solving for x gives $x = 4$, so the plane intersects the x-axis at the point $(4, 0, 0)$.

A similar procedure can be used to determine the intersection with the y and z axes. In particular, this plane intersects the y axis at the point $(0, 3, 0)$, and it intersects the z-axis at the point $(0, 0, 2)$, as shown in Figure 4.

Normal Vectors



▲ **Figure 5:** The vector $\mathbf{n}$ is normal to the plane P.

A **normal vector** to a plane is any vector whose direction is perpendicular to that of the plane, as shown in Figure 5. For example, the vector $(0, 0, 1)$ is normal to any horizontal plane.

There is a close relationship between the linear equation for a plane and the normal vector.

Normal Vector to a Plane

If P is the plane in $\mathbb{R}^3$ defined by the equation

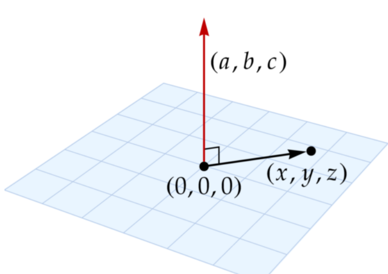
$$ax + by + cz = d,$$

then $\mathbf{n} = (a, b, c)$ is a normal vector for P.

For example, the vector $(3, 4, 2)$ is normal to the plane $3x + 4y + 2z = 15$, and the vector $(1, 0, 1)$ is normal to the plane $x + z = 3$.

We can justify this formula using the dot product. First, consider a plane P that goes through the origin $(0, 0, 0)$. Such a plane has an equation of the form

$$ax + by + cz = 0.$$



▲ **Figure 6:** The plane $ax + by + cz = 0$ consists of all points (x, y, z) whose radial vector is orthogonal to (a, b, c).

Using the dot product, we can rewrite this equation as

$$(a, b, c) \cdot (x, y, z) = 0.$$

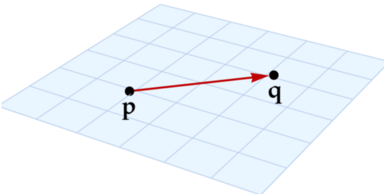
Geometrically, this equation says that the plane P consists of all points (x, y, z) whose radial vector is orthogonal to (a, b, c), as shown in Figure 6. It follows that (a, b, c) is a normal vector for P. Since parallel planes have the same normal vectors, this also holds for any plane of the form $ax + by + cz = d$.

Normal vectors are useful because a normal vector to a plane completely determines the direction of the plane. Indeed, specifying a normal vector is probably the most common way to describe how a plane is oriented in space. Note, however, that the normal vector is not uniquely determined, since any nonzero scalar multiple of a normal vector is again a normal vector.

Parallel Vectors

A vector $\mathbf{v}$ is said to be **parallel** to a given plane if $\mathbf{v}$ can be moved so that its arrow lies entirely on the plane, as shown in Figure 7. Equivalently, $\mathbf{v}$ is parallel to a given plane if there exist points $\mathbf{p}$ and $\mathbf{q}$ on the plane so that $\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Note that any vector parallel to a plane must be orthogonal to the normal vector. Conversely, any vector that is orthogonal to the normal vector must be a parallel vector.



▲ **Figure 7:** A parallel vector to a plane.

EXAMPLE 3

Find the value of t for which the vector $(3, 1, t)$ is parallel to the plane $2x + 4y + 5z = 12$.

SOLUTION The normal vector to this plane is $(2, 4, 5)$. We take the dot product of this with the given vector:

$$(2, 4, 5) \cdot (3, 1, t) \;=\; (2)(3) + (4)(1) + (5)(t) \;=\; 10 + 5t.$$

The given vector will be parallel to the plane when this dot product is zero, which occurs for $\boxed{t = -2}$.

Because the normal vector is orthogonal to all of the parallel vectors, the cross product of any two parallel vectors that point in different directions will yield a normal vector.

Finding an Equation for a Plane

A plane in $\mathbb{R}^3$ can be specified using its normal vector as well as a point on the plane. This is similar to specifying a line in $\mathbb{R}^2$ using the slope of the line as well as a point in the line.

The following example illustrates how to find the equation of a plane from a point and a normal vector.

EXAMPLE 4

Find and equation for the plane through the point $(2, 1, 3)$ that has normal vector $(4, 2, 3)$.

SOLUTION From the normal vector, we know that the plane has an equation of the form

$$4x + 2y + 3z \;=\; d$$

for some constant d. Plugging in the point $(2, 1, 3)$ gives the equation

$$4(2) + 2(1) + 3(3) \;=\; d$$

and thus $d = 19$. Thus one equation for the plane is

$$4x + 2y + 3z \;=\; 19.$$

In the same way that any two points determine a line, any three points determine a plane. More precisely, if $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are three points in $\mathbb{R}^3$ that do not lie on a single line, then there exists a unique plane in $\mathbb{R}^3$ going through all three point.

We say that three points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are **collinear** if there is a line that goes through all three of them. Thus any three non-collinear points determine a plane.

To find the equation of such a plane, observe that the vectors $\mathbf{v} = \mathbf{q} - \mathbf{p}$ and $\mathbf{w} = \mathbf{r} - \mathbf{p}$ are parallel to the plane, and therefore the cross product $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is a normal vector.

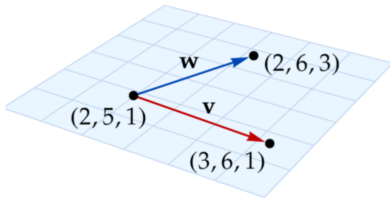


Figure 8: The plane from Example 5.

EXAMPLE 5

Find the equation of the plane that goes through the points $(2,5,1)$, $(3,6,1)$, and $(2,6,3)$.

SOLUTION Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors shown in Figure 8. Then

$$\mathbf{v} = (3,6,1) - (2,5,1) = (1,1,0) \qquad \text{and} \qquad \mathbf{w} = (2,6,3) - (2,5,1) = (0,1,2).$$

Both of these vectors are parallel to the plane, so their cross product is a normal vector:

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = (2,-2,1).$$

Note that $(2,-2,1)$ is orthogonal to both $(1,1,0)$ and $(0,1,2)$.

Then the plane has an equation of the form

$$2x - 2y + z = d$$

for some constant d. Plugging in any one of the three points gives $d = -5$, so one equation for the plane is $\boxed{2x - 2y + z = -5}$.

It is easy to check that all three of the given points satisfy this equation.

Distance from a Point to a Plane

Given a plane in $\mathbb{R}^3$ and a point $\mathbf{p}$ not on the plane, there is always exactly one point $\mathbf{q}$ on the plane that is closest to $\mathbf{p}$, as shown in Figure 9. The point $\mathbf{q}$ is known as the **projection of p onto the plane**, and the distance from $\mathbf{p}$ to $\mathbf{q}$ is the **distance from the point p to the plane**.

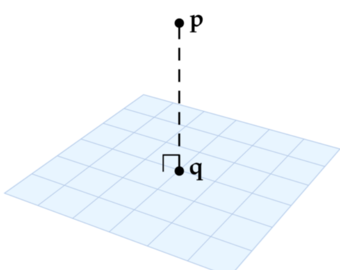


Figure 9: The point $\mathbf{q}$ is the projection of the point $\mathbf{p}$ onto this plane.

We can use the dot product to find the distance from a point $\mathbf{p}$ to a plane. The trick is to first choose *any* point $\mathbf{r}$ that lies on the plane. Then the point $\mathbf{p}$, its projection $\mathbf{q}$, and the point $\mathbf{r}$ make a right triangle, as shown in Figure 10. In this case, the distance from $\mathbf{p}$ to $\mathbf{q}$ is given by the formula

$$\boxed{|\mathbf{s}| = |\mathbf{h} \cdot \mathbf{u}|}$$

where $\mathbf{u}$ is a unit vector normal to the plane.

Here the absolute value is necessary in the case where $\mathbf{u}$ and $\mathbf{s}$ point in opposite directions.

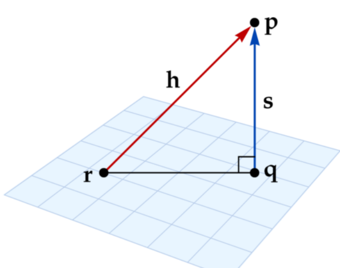


Figure 10: The right triangle made by a point $\mathbf{p}$, its projection $\mathbf{q}$, and another point $\mathbf{r}$ on the plane.

EXAMPLE 6

Find the distance from the point $\mathbf{p} = (8,0,9)$ to the plane $3x - 2y + 4z = 2$.

SOLUTION We start by choosing any point $\mathbf{r}$ on the plane, i.e. any values of x, y, and z that satisfy the given equation. There are many possible choices, but let's use $\mathbf{r} = (0,1,1)$. Then

$$\mathbf{h} = \mathbf{p} - \mathbf{r} = (8,0,9) - (0,1,1) = (8,-1,8).$$

The vector $\mathbf{n} = (3,-2,4)$ is normal to the plane, so a unit normal vector is

$$\mathbf{u} = \frac{1}{|\mathbf{n}|}\mathbf{n} = \frac{1}{\sqrt{29}}(3,-2,4).$$

Then

$$\mathbf{h} \cdot \mathbf{u} = (8,-1,8) \cdot \frac{1}{\sqrt{29}}(3,-2,4) = \frac{58}{\sqrt{29}} = 2\sqrt{29}.$$

so the distance is $\boxed{2\sqrt{29}}$.

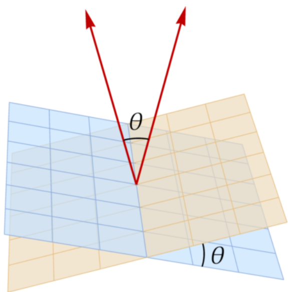


Figure 11: The angle between two planes is the same as the angle between the two normal vectors.

Angle Between Planes

Two planes that intersect form an angle, sometimes called a **dihedral angle**. As Figure 11 illustrates, the angle between two planes is the same as the angle between their corresponding normal vectors.

Of course, there are two possible directions for each normal vector, which are opposite from one another. There are also two different angles between the planes, namely the acute angle θ shown in Figure 11, and the obtuse angle $180° - \theta$. Depending on which pair of normal vectors we choose, the angle between them might be either θ or $180° - \theta$.

EXAMPLE 7

Find the angle between the planes $x + z = 1$ and $x + y + z = 2$.

SOLUTION The corresponding normal vectors are $\mathbf{v} = (1, 0, 1)$ and $\mathbf{w} = (1, 1, 1)$. The formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$ gives

$$2 = \sqrt{2}\sqrt{3}\cos\theta.$$

Then

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35.26°.$$

If we had switched the direction of one the normal vectors, such as using

$$\mathbf{w} = (-1, -1, -1),$$

then the angle between $\mathbf{v}$ and $\mathbf{w}$ would have been

$$180° - 35.26° = 144.74°.$$

Note that each plane has normal vectors in two possible directions, which are opposite from one another.

Hyperplanes

A linear equation in four variables has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b,$$

where a_1, a_2, a_3, a_4, and b are constants. Such an equation defines a **hyperplane** in $\mathbb{R}^4$. A hyperplane is similar to a plane, except that it is three-dimensional. That is, in the same way that a plane is like a copy of $\mathbb{R}^2$ sitting inside of $\mathbb{R}^3$, a hyperplane is like a copy of $\mathbb{R}^3$ sitting inside of $\mathbb{R}^4$.

Here are some examples of linear equations in $\mathbb{R}^4$ and the corresponding hyperplanes:

- The equation $x_4 = 0$ defines the $x_1x_2x_3$-hyperplane in $\mathbb{R}^4$, i.e. the hyperplane that contains the x_1, x_2, and x_3 axes.
- If b is any constant, the equation $x_4 = b$ defines a hyperplane in $\mathbb{R}^4$ that is parallel to the $x_1x_2x_3$-hyperplane.
- The equations $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ define the $x_2x_3x_4$-hyperplane, the $x_1x_3x_4$-hyperplane, and the $x_1x_2x_4$-hyperplane, respectively.
- The equation $x_1 + x_2 + x_3 + x_4 = 1$ defines a slanted hyperplane in $\mathbb{R}^4$, which goes through the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$.

Any vector between two points in a hyperplane is said to be **parallel** to the hyperplane. A vector is **normal** to a hyperplane if it is orthogonal to every parallel vector. As in $\mathbb{R}^3$, the hyperplane in $\mathbb{R}^4$ defined by the equation $a_1x_1 + a_2x_2 + a_3x_3 + a - 4x_4 = b$ has (a_1, a_2, a_3, a_4) as a normal vector.

Here the word **flat** refers to any infinite, boundless shape that does not bend or curve. Lines and planes are examples of flats, but in $\mathbb{R}^n$ a flat may have any number of dimensions from 1 to $n-1$.

In general, the word "hyperplane" refers to an $(n-1)$-dimensional flat in $\mathbb{R}^n$. For example, hyperplanes in $\mathbb{R}^5$ are 4-dimensional flats and hyperplanes in $\mathbb{R}^{17}$ are 16-dimensional flats. Any hyperplane in $\mathbb{R}^n$ can be defined by a linear equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are constants. Such a hyperplane always has $(a_1, a_2, \ldots, a_n)$ as a normal vector.

EXERCISES

1. Find the equation of the plane that is parallel to $x - 3y + 2z = 4$ and goes through the point $(2, 1, 5)$.

2. Find the points at which the plane $4x - 2y + 5z = 20$ intersects the x, y, and z axes.

3–4 ■ Find an equation for the plane through the point $\mathbf{p}$ that has normal vector $\mathbf{n}$.

3. $\mathbf{p} = (-2, 3, 1)$, $\mathbf{n} = (1, -2, 4)$

4. $\mathbf{p} = (3, 4, 2)$, $\mathbf{n} = (1, 0, 2)$

5. Find an equation for the plane that goes through the points $(1, 1, 0)$, $(2, 0, 1)$, and $(3, 1, 3)$.

6. Let L be the line in $\mathbb{R}^3$ that goes through the points $(2, -4, 5)$ and $(3, 0, 7)$. Find an equation for the plane through $(2, -4, 5)$ that is perpendicular to L.

7. Find a vector that is parallel to the plane $-x + 5y - 2z = 3$ and orthogonal to $(1, 1, 2)$.

8. (a) Find the distance from the point $(5, 6, 3)$ to the plane $x + y + z = 2$.

(b) Find the projection of the point $(5, 6, 3)$ onto this plane.

(c) Find the reflection of the point $(5, 6, 3)$ across this plane.

9. Find the distance between the planes $x + 2y - 2z = 2$ and $x + 2y - 2z = 17$.

10–11 ■ Find the angle between the given planes.

10. $x + y + 2z = 3$, $2x - y + z = 1$

11. $x + z = 5$, $3x + 5y - 3z = 7$

12. Find an equation for the hyperplane in $\mathbb{R}^4$ that goes through the point $(2, 1, 5, 2)$ and has normal vector $(1, -1, 1, -1)$.

13. Let H be the hyperplane in $\mathbb{R}^4$ defined by the equation $x_1 + 2x_2 + 2x_3 + 4x_4 = 4$. Find the distance from the point $(4, 5, 6, 7)$ to H.