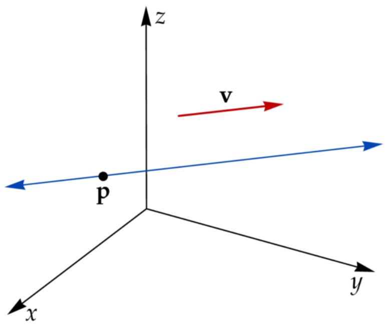
9.1 Lines

As we have seen, a linear equation in x, y, and z describes a plane in $\mathbb{R}^3$. This is quite helpful for understanding planes, but this leaves us without a good description of *lines* in $\mathbb{R}^3$. In this section, we develop a description of lines in $\mathbb{R}^3$ (and more generally $\mathbb{R}^n$) using a new kind of equation called a **parametric equation**.

The Parametric Equation for a Line



▲ **Figure 1:** A line in $\mathbb{R}^3$.

Figure 1 shows a typical line in $\mathbb{R}^3$. This line goes through a certain point $\mathbf{p}$, and is parallel to a certain vector $\mathbf{v}$. For the line in the picture, these are

$$\mathbf{p} = (4,1,3) \qquad \text{and} \qquad \mathbf{v} = (-2,2,0).$$

What are some other points on this line?

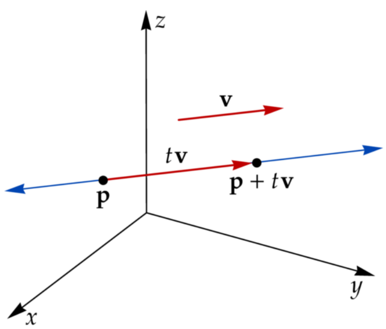
Well, if we add $\mathbf{p}$ to $\mathbf{v}$, we certainly get another point on the line:

$$\mathbf{p} + \mathbf{v} = (2,3,3).$$

More generally, if t is any scalar, then the vector $t\mathbf{v}$ is also parallel to the line, so

$$\mathbf{p} + t\mathbf{v} = (4,1,3) + t(-2,2,0) = (4-2t, 1+2t, 3)$$

will be another point on the line. Figure 2 shows the location of this point for $t = 1.5$.



▲ **Figure 2:** The point $\mathbf{p} + t\mathbf{v}$ lies on the line for any scalar t.

The equation

$$(x,y,z) = (4-2t, 1+2t, 3)$$

is called a **parametric equation** (or **parametrization**) for the line. This equation includes an arbitrary scalar t (called a **parameter**), and we can plug in any value for t to get a point (x,y,z) on the line. For example:

- Substituting $t = 1.5$ gives the point $(1,4,3)$,
- Substituting $t = -1$ gives the point $(6,-1,3)$, and
- Substituting $t = 0$ gives the original point $(4,1,3)$.

In fact, every point on the line corresponds to some value of t.

Parametric Equation for a Line

Let L be the line in $\mathbb{R}^3$ that goes through a point $\mathbf{p}$ and is parallel to a vector $\mathbf{v}$. Then L is defined by the parametric equation

$$(x,y,z) = \mathbf{p} + t\mathbf{v}.$$

If $\mathbf{p} = (x_0, y_0, z_0)$ and $\mathbf{v} = (a,b,c)$, this equation can be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{bmatrix}.$$

Note then that the constant terms are the coordinates of a point on the line, and the coefficients of t are the components of the parallel vector. For example, the line

It is usually cleaner to write the parametric equation for a line using column vectors.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+4t \\ 7-3t \\ 5+2t \end{bmatrix}$$

goes through the point $(2,7,5)$ and is parallel to the vector $(4,-3,2)$.

Note that we can also find a parametric equation for the line through two points, as the following example shows.

EXAMPLE 1

Find a parametric equation for the line through the points $(3,7,4)$ and $(5,8,2)$.

SOLUTION Let $\mathbf{v}$ be the vector that goes between these two points:

$$\mathbf{v} = (5,8,2) - (3,7,4) = (2,1,-2).$$

Then $\mathbf{v}$ should be parallel to the line. We can use either of the two given points to find a parametric equation. Using $(3,7,4)$, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

If we had used $(5,8,2)$ instead, we would have found a different parametric equation for the same line.

or equivalently

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3+2t \\ 7+t \\ 4-2t \end{bmatrix}.$$

It is not always necessary to use a point and a direction vector to make up a parametric equation for a line. For example, it is obvious that

$$(x,y,z) = (t,0,0)$$

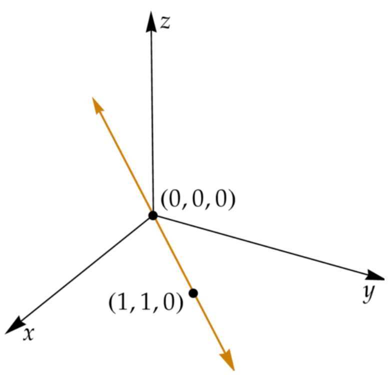
is a parametric equation for the x-axis in $\mathbb{R}^3$, and similar parametric equations can be made for the y and z axes. Some other simple examples of lines include:

- Any line parallel to the x-axis is defined by an equation of the form $(x,y,z) = (t,y_0,z_0)$, where y_0 and z_0 are constants. Similar equations define lines parallel to the y and z axes.

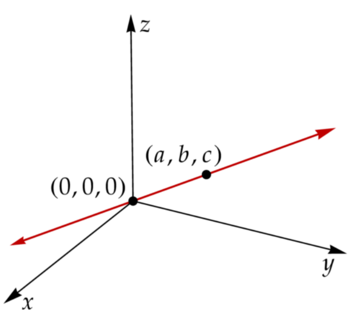
- The line of all points on the xy-plane for which $y = x$ can be described parametrically as $(x,y,z) = (t,t,0)$. This line is shown in Figure 3

- The line $(x,y,z) = (t,t,t)$ goes through the origin and the point $(1,1,1)$. Each point on this line is equidistant from the x, y, and z axes.

- More generally, if (a,b,c) is any point in $\mathbb{R}^3$, then $(x,y,z) = t(a,b,c)$ is the line of all scalar multiples of (a,b,c). This line goes through the point (a,b,c) as well as the origin, as shown in Figure 4.



▲ **Figure 3:** The line $(x,y,z) = (t,t,0)$.



▲ **Figure 4:** The line $(x,y,z) = t(a,b,c)$.

Testing Points

A parametric equation is a very different way of describing a shape than a Cartesian equation. When we write a Cartesian equation for a plane, such as

$$2x + 5y + 3z = 4,$$

the equation is actually a *test* for whether a given point lies on the plane. For example, the point $(-2, 1, 1)$ lies on this plane, since substituting $x = -2$, $y = 1$, and $z = 1$ makes the equation true.

A parametric equation, on the other hand, is more like a *factory* for producing points on the plane. Given a parametric equation for a line such as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3 \\ 8 \\ -1 \end{bmatrix}$$

we can plug in values for t to *make* points on the line.

The nature of parametric equations makes them very useful for computer graphics. For example, if we want a computer to draw a line for us, all it has to do is plug in 1000 different values of t to get 1000 different points on the line, and then plot all of them to get a good picture.

However, we can still use a parametric equation to test whether a given point lies on a given line.

EXAMPLE 2

Determine whether the point $(8, 1, 2)$ lies on the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+t \\ 5-t \\ 3-2t \end{bmatrix}$.

SOLUTION We are looking for a value of t for which

$$\begin{bmatrix} 4+t \\ 5-t \\ 3-2t \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 2 \end{bmatrix}.$$

This gives us three equations involving t:

$$4 + t = 8, \qquad 5 - t = 1, \qquad 3 - 2t = 2.$$

From the first equation, we see that $t = 4$ is the only possibility. This works for the second equation, but not for the third, which means that $(8, 1, 2)$ does not lie on the given line.

To solve a system of equations with one unknown, solve the first equation and then check whether the solution works for all of the other equations.

Lines in $\mathbb{R}^n$

Of course, our description of lines in $\mathbb{R}^3$ works just as well in higher dimensions.

Parametric Equation for a Line in $\mathbb{R}^n$

Let L be the line in $\mathbb{R}^n$ that goes through a point $\mathbf{p}$ and is parallel to a vector $\mathbf{v}$. Then L is defined by the parametric equation

$$(x_1, x_2, \ldots, x_n) = \mathbf{p} + t\mathbf{v}.$$

For example, the line in $\mathbb{R}^4$ that goes through the point $(1, 0, 1, 0)$ in the direction of the vector $(0, 1, 1, 2)$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

or equivalently $(x_1, x_2, x_3, x_4) = (1, t, 1 + t, 2t)$.

Again, it is sometimes possible to make a parametric equation for a line in $\mathbb{R}^4$ without using a point or direction vector. For example, the x_4-axis in $\mathbb{R}^5$ is defined by the parametric equation

$$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, t, 0).$$

EXAMPLE 3

Find a parametric equation for the line in $\mathbb{R}^5$ that goes through the points $(2, 1, 4, 7, 6)$ and $(5, 3, 3, 4, 8)$.

SOLUTION Let $\mathbf{v}$ be the vector that goes between these two points:

$$\mathbf{v} = (5, 3, 3, 4, 8) - (2, 1, 4, 7, 6) = (3, 2, -1, -3, 2).$$

Then $\mathbf{v}$ should be parallel to the line. Using this vector and the first point gives us the equation

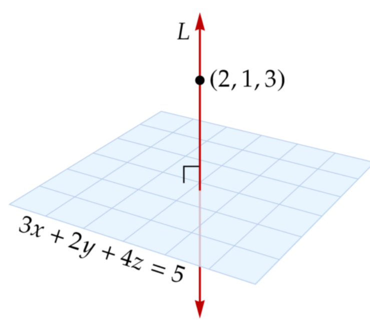
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 7 \\ 6 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \\ -3 \\ 2 \end{bmatrix}.$$

EXERCISES

1. Find a parametric equation for the line in $\mathbb{R}^3$ that goes through the point $(5, 7, 8)$ and is parallel to the vector $(4, 1, 7)$.

2. Find a vector that is parallel to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + 3t \\ 5 - 2t \\ 4 + t \end{bmatrix}$.

3. In the following figure, the line L is perpendicular to the plane $3x + 2y + 4z = 5$ and goes through the point $(2, 1, 3)$.



Find a parametric equation for L.

4. Find a parametric equation for the line through $(3, 1, 0)$ that is parallel to the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 + t \\ 3 - 2t \\ 5 + 4t \end{bmatrix}.$$

5. Find a parametric equation for the line in $\mathbb{R}^3$ that goes through the points $(2, 1, 4)$ and $(3, 6, 2)$.

6. What angle does the line $(x, y, z) = (t, t, t)$ make with the x-axis?

7. Let L be the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + 3t \\ 8 - t \\ 5 + 2t \end{bmatrix}.$$

Which of the points $(5, 7, 6)$, $(8, 6, 9)$, and $(-1, 4, 3)$ lies on L?

8. Let L_1 and L_2 be the lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 + t \\ 8 + 2t \\ 1 - t \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - 2t \\ 6 - 4t \\ 2t \end{bmatrix}.$$

Is one of these the same as the line $(x, y, z) = (1 + t, 2t, 3 - t)$?

9. Find a parametric equation for the line $y = 2x + 3$ in $\mathbb{R}^2$.

10. Find a parametric equation for the line in $\mathbb{R}^4$ that goes through the point $(2, 1, 5, 3)$ and is parallel to the x_2-axis.

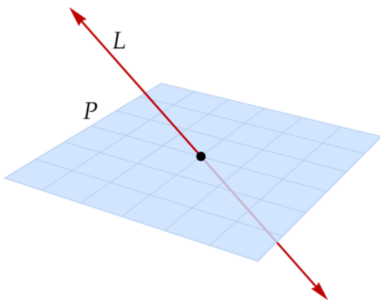
11. Find a parametric equation for the line in $\mathbb{R}^4$ that goes through the origin and is perpendicular to the hyperplane $2x_1 - x_2 + 4x_3 + x_4 = 1$.

9.2 Geometry of Lines

In this section we use parametric equations to explore the geometry of lines in $\mathbb{R}^3$.

Line-Plane Intersections

Typically, a line and a plane in $\mathbb{R}^3$ intersect at exactly one point, as shown in Figure 1. The following example shows how to find this point of intersection.



▲ **Figure 1:** A line L and plane P in $\mathbb{R}^3$ typically intersect at one point.

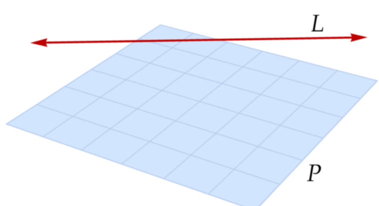
EXAMPLE 1

Find the point at which the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-t \\ 2t-3 \\ t \end{bmatrix}$ intersects the plane $x + y + z = 4$.

SOLUTION What we want is a value of t for which the point $(1-t, 2t-3, t)$ lies on the given plane. We can test this by plugging the point into the equation for the plane:

$$(1-t) + (2t-3) + (t) = 4.$$

Simplifying and solving for t yields $t = 3$, and plugging this back into the equation of the line gives the intersection point, which is $\boxed{(-2, 3, 3)}$.



▲ **Figure 2:** A line and plane in $\mathbb{R}^3$ that are parallel.

Of course, it is also possible for a line and plane in $\mathbb{R}^3$ to be **parallel**, meaning that there is no point of intersection. For example, if P_1 and P_2 are parallel planes, then any line drawn on P_2 is parallel to P_1. Figure 2 shows an example of a line that is parallel to a plane.

When a line and plane are parallel, searching for the point of intersection yields a contradictory equation. For example, the line $(x, y, z) = (t, t, t)$ is parallel to the plane $x + y - 2z = 1$, and if we plug the point (t, t, t) into the equation $x + y - 2z = 1$ we get

$$t + t - 2t = 1.$$

This simplifies to the equation $0 = 1$, which indicates that there are no solutions for t.

An equation is said to be a **tautology** if it is true for any values of the variables.

Finally, it is possible for a line to lie on a plane, meaning that every point on the line also lies on the plane. For such a line, searching for the point of intersection yields a tautological equation. For example, the line $(x, y, z) = (t, t, t)$ lies on the plane $x + y - 2z = 0$, and if we plug the point (t, t, t) into the equation $x + y - 2z = 0$ we get

$$t + t - 2t = 0.$$

which is true for every value of t.

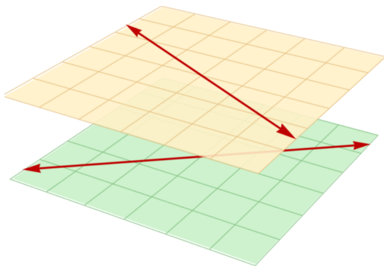
Pairs of Lines

As we know from plane geometry, two lines in a plane are either parallel or intersect at single point. Parallel lines go in the same direction, maintaining a constant distance between them, and each line can be moved onto to the other by a translation.

In three dimensions, there are *three* possible relationships between a pair of lines:

1. They can be parallel, meaning that they go in the same direction. Parallel lines maintain a constant distance between one another, and each line can be moved onto the other by a translation.

2. They can intersect at a single point.



▲ **Figure 3:** A pair of skew lines in $\mathbb{R}^3$.

3. They can be **skew**, as shown in Figure 3. Skew lines do not go in the same direction, but also do not intersect.

Any two parallel or intersecting lines in $\mathbb{R}^3$ lie on a common plane, but skew lines do not. However, every pair of skew lines do lie on a uniquely determined pair of parallel planes, as shown in Figure 3.

But how can we determine whether two given lines are parallel, intersecting, or skew? Well first of all, two lines are parallel if and only if their direction vectors are scalar multiples of one another. For example, the lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} + t\begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}.$$

are parallel, since the direction vectors $(-2, 4, -6)$ and $(3, -6, 9)$ are scalar multiples of one another:

$$(3, -6, 9) = -\frac{3}{2}(-2, 4, -6).$$

For lines that are not parallel, a simple way of checking whether they intersect is to look for a point of intersection. The following example illustrates this procedure.

EXAMPLE 2

Find the point at which the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6+t \\ 3+3t \\ 2+2t \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+2t \\ 6+t \\ 9-t \end{bmatrix}$ intersect.

SOLUTION We are looking for a point that lies on both lines, and there is no reason that the two values of t corresponding to this point would be the same. Thus we want to find values t_1 and t_2 for t so that

$$\begin{bmatrix} 6+t_1 \\ 3+3t_1 \\ 2+2t_1 \end{bmatrix} = \begin{bmatrix} 2+2t_2 \\ 6+t_2 \\ 9-t_2 \end{bmatrix}.$$

This gives us three equations involving t_1 and t_2:

$$6+t_1 = 2+2t_2, \qquad 3+3t_1 = 6+t_2, \qquad 2+2t_1 = 9-t_2.$$

A system of equations with more equations than unknowns is called an **overdetermined system**. This system has three equations and two unknowns, which makes it overdetermined.

To look for a solution to these three equations, we treat the first two as a system of two equations and two unknowns. Solving this system for t_1 and t_2 in the usual way yields

$$t_1 = 2 \qquad \text{and} \qquad t_2 = 3.$$

To solve a system of three equations with two unknowns, start by solving the first two equations and then check whether your answer works for the third equation.

We now plug these values into the third equation to check whether this is actually a solution:

$$2+2t_1 = 9-t_2 \qquad \rightarrow \qquad 2+2(2) = 9-(3) \ \checkmark$$

Since the equation on the right is true, we have found a point of intersection. Substituting back into the parametric equation for one of the lines gives the point $\boxed{(8, 9, 6)}$.

This procedure always finds an intersection point if there is one. If the three equations involving t_1 and t_2 have no solution, it means that the lines are skew.

A Closer Look Coplanar Lines

Two lines are said to be **coplanar** if they lie on a common plane. Intersecting lines are always coplanar, as are parallel lines, but skew lines are not.

There is a simple test using determinants to check whether two lines in $\mathbb{R}^3$ are coplanar. If L_1 and L_2 are lines in $\mathbb{R}^3$ with parametric equations

$$(x, y, z) = \mathbf{p} + t\mathbf{v} \qquad \text{and} \qquad (x, y, z) = \mathbf{q} + t\mathbf{w},$$

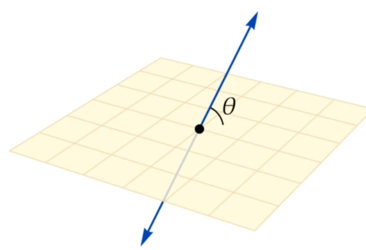
then L_1 and L_2 are coplanar if and only if the following determinant is zero.

$$\begin{vmatrix} v_x & v_y & v_z \\ w_x & w_y & w_z \\ p_x - q_x & p_y - q_y & p_z - q_z \end{vmatrix}$$

The reason is that if L_1 and L_2 are coplanar, then the vectors $\mathbf{v}$ and $\mathbf{w}$ must both be parallel to this plane, as must the vector $\mathbf{p} - \mathbf{q}$. That is, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p} - \mathbf{q}$ must be coplanar vectors, which means that the determinant above is zero. Conversely, if the determinant above is zero, then the vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p} - \mathbf{q}$ must be coplanar, and it follows that the two lines are coplanar.

EXERCISES

1. The following figure shows the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 + 2t \\ 2 - 5t \\ 5t \end{bmatrix}$ and the plane $x + 2y + 7z = -2$.



(a) Find the point of intersection.

(b) Find the angle θ.

2. Where does the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 - t \\ t \end{bmatrix}$ intersect the plane $x + y + z = 1$?

3. Find the point in $\mathbb{R}^4$ at which the line $(x_1, x_2, x_3, x_4) = (t, t, t, t)$ intersects the hyperplane $x_1 + 2x_2 + x_3 + 2x_4 = 24$.

4–6 ■ Determine whether the given lines are parallel, intersecting, or skew. If they intersect, find the point of intersection.

4. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ 3 \\ 1 - t \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - 4t \\ 1 \\ 2t \end{bmatrix}$.

5. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3+2t \\ 3+t \\ 4+3t \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3-t \\ t \\ 5-2t \end{bmatrix}$.

6. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ 1-t \\ t \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 6-3t \\ 1+t \end{bmatrix}$.

7. Find the point at which the lines

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ t \\ 5 \\ 1+t \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t \\ 3 \\ 2+3t \\ 3+t \end{bmatrix}$$

intersect in $\mathbb{R}^4$.

8. Let L_1 and L_2 be the skew lines $(x, y, z) = (t, 3, 2)$ and $(x, y, z) = (1, t, 5)$.

(a) Find a parallel pair of planes that contain L_1 and L_2.

(b) Find a line L_3 that intersects L_1 and L_2 and is perpendicular to both of them.

9. Find an equation for the plane in $\mathbb{R}^3$ that contains the lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

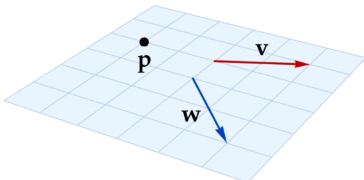
10. The planes $x = 3$ and $2x + 3y + 4z = 13$ intersect along a line in $\mathbb{R}^3$. Find a parametric equation for this line.

11. Find the distance from the point $(6, 1, 1)$ to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ 2+t \\ 3-t \end{bmatrix}$.

9.3 Planes and Flats

So far we have learned how to use parametric equations to describe lines in $\mathbb{R}^n$. In this section, we generalize this technique to allow for descriptions of planes and other flats.

Parametric Equations for Planes



▲ **Figure 1:** A plane in $\mathbb{R}^n$.

Figure 1 shows a typical plane in $\mathbb{R}^n$. This plane goes through a certain point $\mathbf{p}$, and is parallel to the vectors $\mathbf{v}$ and $\mathbf{w}$. We assume that $\mathbf{v}$ and $\mathbf{w}$ do not point in the same direction, so that $\mathbf{v}$ and $\mathbf{w}$ together specify the direction of the plane. How can we describe the points on this plane?

Well, since $\mathbf{v}$ and $\mathbf{w}$ are parallel to the plane, any linear combination

$$s\mathbf{v} + t\mathbf{w}$$

is also parallel to the plane. Adding this vector to $\mathbf{p}$, we conclude that any point of the form

$$\mathbf{p} + s\mathbf{v} + t\mathbf{w}$$

lies on the plane, where s and t can be any scalars. This is the desired parametric equation for the plane.

Parametric Equation for a Plane

Let P be a plane in $\mathbb{R}^n$ determined by a point $\mathbf{p}$ and parallel vectors $\mathbf{v}$ and $\mathbf{w}$. Then P is defined by the parametric equation

$$(x_1, x_2, \ldots, x_n) = \mathbf{p} + s\mathbf{v} + t\mathbf{w}$$

where s and t are parameters.

Here both s and t are parameters, meaning that each of them can be an arbitrary real number.

For example, the plane in $\mathbb{R}^3$ that goes through the point $(3, 1, 4)$ and is parallel to the vectors $(1, 3, 1)$ and $(1, 1, 2)$ is defined by the equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + s \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 + s + t \\ 1 + 3s + t \\ 4 + s + 2t \end{bmatrix}.$$

Note that the coefficients of s on the right are the components of a parallel vector, as are the coefficients of t.

Of course, we already know how to describe planes in $\mathbb{R}^3$ using linear Cartesian equations, which makes parametric equations for planes a bit redundant. However, in higher dimensions, a linear Cartesian equation describes a hyperplane instead of a plane, so we need parametric equations to describe planes in $\mathbb{R}^n$ for $n > 3$.

EXAMPLE 1

Find a parametric equation for the plane in $\mathbb{R}^4$ that goes through the points $(1, 1, 1, 1)$, $(1, 2, 3, 4)$, and $(4, 3, 2, 1)$.

SOLUTION We can get some vectors parallel to the plane by subtracting pairs of points:

$$\mathbf{v} = (1, 2, 3, 4) - (1, 1, 1, 1) = (0, 1, 2, 3), \qquad \mathbf{w} = (4, 3, 2, 1) - (1, 1, 1, 1) = (3, 2, 1, 0).$$

Note here that the three given points correspond to certain values of s and t:

point	s	t
$(1,1,1,1)$	0	0
$(1,2,3,4)$	1	0
$(4,3,2,1)$	0	1

Then the given plane is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1+3t \\ 1+s+2t \\ 1+2s+t \\ 1+3s \end{bmatrix}.$$

Most of the techniques we know for working with parametric equations for lines are also useful for planes.

EXAMPLE 2

Determine whether the point $(1,4,4,10)$ lies on the plane

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ 1-s+3t \\ -3+s+2t \\ 2+2s+t \end{bmatrix}.$$

SOLUTION We want to find values for s and t such that

$$\begin{bmatrix} s-t \\ 1-s+3t \\ -3+s+2t \\ 2+2s+t \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 10 \end{bmatrix}.$$

Remember the procedure for solving an overdetermined system:

1. Use the first few equations to solve for the variables.
2. Check whether the solution works for the remaining equations.

This gives us a system of four equations with two unknowns:

$$s-t = 1, \qquad 1-s+3t = 4, \qquad -3+s+2t = 4, \qquad 2+2s+t = 10.$$

Solving the first two equations for s and t gives $s=3$ and $t=2$. This solution works for the other two equations as well, and therefore the point $(1,4,4,10)$ does lie on the given plane.

By the way, in simple examples it is often possible to figure out a parametric equation for a plane without worrying about parallel vectors and such. Here are some examples:

1. The xy plane in $\mathbb{R}^3$ is defined by the equation $(x,y,z) = (s,t,0)$. Similar equations work for the xz plane and the yz plane.

2. Similarly, in $\mathbb{R}^4$ the equation $(x_1,x_2,x_3,x_4) = (0,s,0,t)$ defines the x_2x_4 plane, i.e. the plane that contains the x_2 and x_4 axes. Similar equations work for the other five coordinate planes.

There are *six* coordinate planes in $\mathbb{R}^4$, namely the x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, and x_3x_4 planes.

3. The plane $x=3$ in $\mathbb{R}^3$ is defined by the equation $(x,y,z) = (3,s,t)$.

4. The plane $x=z$ in $\mathbb{R}^3$ is defined by the equation $(x,y,z) = (s,t,s)$.

5. The plane $z=2y$ in $\mathbb{R}^3$ is defined by the equation $(x,y,z) = (s,t,2t)$.

6. The plane $y+z=1$ in $\mathbb{R}^3$ is defined by the equation $(x,y,z) = (s,t,1-t)$.

Parametric Equations for Flats

The fundamental shapes in two-dimensional geometry are lines, and the fundamental shapes in three-dimensional geometry are lines and planes. In higher dimensions, the fundamental shapes include lines and planes, as well as higher-dimensional analogs of these. Collectively, these shapes are known as **flats**.

Every flat has a **dimension**, and any two flats of the same dimension are geometrically congruent. A one-dimensional flat is the same as a line, and a two-dimensional flat is the same as a plane. However, there are also three-dimensional flats, four-dimensional flats, and so forth.

In addition, it sometimes helps to think of points as **zero-dimensional flats**.

Each of the Euclidean spaces

$$\mathbb{R}, \quad \mathbb{R}^2, \quad \mathbb{R}^3, \quad \mathbb{R}^4, \quad \mathbb{R}^5, \quad \ldots$$

is itself a flat. For example, the real numbers $\mathbb{R}$ form a line (the **real line**), and $\mathbb{R}^2$ is a plane (the **Euclidean plane**). Similarly, $\mathbb{R}^3$ is a three-dimensional flat, $\mathbb{R}^4$ is a four-dimensional flat, and so forth.

We can parametrize arbitrary flats in much the same way that we parametrize planes. For example, the parametric equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

defines a 3-dimensional flat in $\mathbb{R}^4$. Here s, t, and u are the three parameters, and we can obtain every point on the 3-dimensional flat by plugging in values for s, t, and u. In particular, $(2,1,5,3)$ is a point on this flat, and the vectors $(1,1,0,0)$, $(0,1,1,0)$, and $(0,0,1,1)$ are **parallel vectors** for this flat.

For flats of dimension four or more, we usually use $t_1, t_2, \ldots$ for parameters instead of separate letters. For example, the parametric equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

defines a 4-dimensional flat in $\mathbb{R}^6$. This flat goes through the point $(1,1,1,1,1,1)$, and is parallel to the vectors

$$(1,0,2,0,0,0), \qquad (0,1,0,0,0,0), \qquad (0,0,0,1,0,3), \qquad \text{and} \qquad (0,0,0,0,1,4).$$

Parametric Equation for a Flat

Let F be a k-dimensional flat in $\mathbb{R}^n$ determined by a point $\mathbf{p}$ and parallel vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then F is defined by the parametric equation

$$(x_1, x_2, \ldots, x_n) = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k$$

where $t_1, t_2, \ldots, t_k$ are parameters.

EXAMPLE 3

Find a parametric equation for the 3-dimensional flat in $\mathbb{R}^4$ that goes through the points $(1,2,1,2)$, $(1,2,3,4)$, $(3,3,3,3)$, and $(4,3,2,1)$.

SOLUTION We can subtract the first point from the remaining points to obtain three parallel vectors:

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} - \begin{bmatrix}1\\2\\1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\2\\2\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}3\\3\\3\\3\end{bmatrix} - \begin{bmatrix}1\\2\\1\\2\end{bmatrix} = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}4\\3\\2\\1\end{bmatrix} - \begin{bmatrix}1\\2\\1\\2\end{bmatrix} = \begin{bmatrix}3\\1\\1\\-1\end{bmatrix}.$$

Then the flat is defined by the parametric equation

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}1\\2\\1\\2\end{bmatrix} + s\begin{bmatrix}0\\0\\2\\2\end{bmatrix} + t\begin{bmatrix}2\\1\\2\\1\end{bmatrix} + u\begin{bmatrix}3\\1\\1\\-1\end{bmatrix}.$$

Note here that the four given points correspond to certain values of s, t, and u:

point	s	t	u
$(1,2,1,2)$	0	0	0
$(1,2,3,4)$	1	0	0
$(3,3,3,3)$	0	1	0
$(4,3,2,1)$	0	0	1

Here are some simple examples of higher-dimensional flats:

1. There is a three-dimensional flat in $\mathbb{R}^5$ that contains the x_1, x_3, and x_4 axes. This is defined by the parametric equation
$$(x_1, x_2, x_3, x_4) = (s, 0, t, u, 0).$$

2. The equation $x_2 = 5$ defines a hyperplane in $\mathbb{R}^5$, which is a four-dimensional flat. This flat can be defined by the parametric equation
$$(x_1, x_2, x_3, x_4, x_5) = (t_1, 5, t_2, t_3, t_4).$$
This flat is parallel to the four-dimensional flat that contains the x_1, x_3, x_4, and x_5 axes.

In general, a hyperplane in $\mathbb{R}^n$ is an $(n-1)$-dimensional flat.

3. The linear equation $x_4 = 2x_2$ defines a hyperplane in $\mathbb{R}^4$, which is a three-dimensional flat. This flat can be defined by the parametric equation
$$(x_1, x_2, x_3, x_4) = (s, t, u, 2t).$$

4. The set of all points $(x_1, x_2, x_3, x_4, x_5, x_6)$ in $\mathbb{R}^6$ for which
$$x_1 = x_2, \quad x_3 = x_4, \quad \text{and} \quad x_5 = x_6$$
is a three-dimensional flat in $\mathbb{R}^6$. It can be defined by the parametric equation
$$(x_1, x_2, x_3, x_4, x_5, x_6) = (s, s, t, t, u, u).$$

EXERCISES

1. Find a parametric equation for the plane in $\mathbb{R}^3$ that goes through the point $(1,2,5)$ and is parallel to the plane
$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3+s\\1+2t\\s-t\end{bmatrix}.$$

2. Find a Cartesian equation for the plane

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+s \\ 2+2s+3t \\ 1+t \end{bmatrix}.$$

3. Find a parametric equation for the plane in $\mathbb{R}^4$ that goes through the points $(1,1,1,1)$, $(2,2,1,1)$, and $(1,1,2,3)$.

4. Find a parametric equation for the plane in $\mathbb{R}^5$ consisting of all points (x_1,x_2,x_3,x_4,x_5) for which

$$x_1 = x_3 = x_5 \qquad \text{and} \qquad x_2 = x_4.$$

5. Let P be the plane

$$(x_1,x_2,x_3,x_4) = (s,t,1+s,2t).$$

Which of the points $(2,4,3,8)$, $(3,5,4,2)$, and $(1,3,4,6)$ lies on P?

6. Find a parametric equation for the plane in $\mathbb{R}^4$ that contains the lines

$$(x_1,x_2,x_3,x_4) = (3,t,1,2) \qquad \text{and} \qquad (x_1,x_2,x_3,x_4) = (3,5,1,t).$$

7. Find the point on the plane

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} s \\ 1+2s \\ t \\ s-t \\ -2+3t \end{bmatrix}$$

whose x_2-coordinate is 9 and whose x_4-coordinate is 1.

8. Find the point in $\mathbb{R}^4$ at which the planes

$$(x_1,x_2,x_3,x_4) = (s,t,6,s) \qquad \text{and} \qquad (x_1,x_2,x_3,x_4) = (4,s,2s,t)$$

intersect.

9. Find a parametric equation for the three-dimensional flat in $\mathbb{R}^4$ that goes through the points $(0,0,0,0)$, $(1,0,0,1)$, $(0,1,0,2)$, and $(0,0,1,-1)$.

10. Find a parametric equation for the three-dimensional flat in $\mathbb{R}^5$ consisting of all points (x_1,x_2,x_3,x_4,x_5) for which

$$x_1 + x_2 = 1 \qquad \text{and} \qquad x_3 = x_5.$$

11. Find a parametric equation for the three-dimensional flat in $\mathbb{R}^5$ that contains the lines

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1+t \\ 1+t \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1+t \\ 1-t \end{bmatrix}.$$

12. The intersection of the plane

$$(x_1, x_2, x_3, x_4) = (3, s, t, 4t)$$

and the hyperplane

$$(x_1, x_2, x_3, x_4) = (s, 2s, t, u)$$

is a line in $\mathbb{R}^4$. Find a parametric equation for this line.