Algebraic slice spectral sequences

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(joint work with Dominic Culver and Hana Jia Kong)

Outline

Motivation and main theorems

Theorem C: Comparison square

Theorem D: *kgl* is algebraically sliceable

Theorem B: ESSS for *kgl*

Theorem A for $k\mathbb{R}$

Future work

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Motivation

Equivariant

- (Hill–Hopkins–Ravenel) For $E \in \mathsf{SH}(G)$, HHR slice spectral sequence
$$E_1^{m,q,n} = \pi^G_{m,n} P^q_q E \Rightarrow \pi^G_{m,n} E.$$

- (HHR) Kervaire invariant one problem.
- Chromatic homotopy theory.

Examples

- (Hill–Hopkins–Ravenel; Hu–Kriz; Araki) $G = C_2$, $E = M\mathbb{R}$ and $E = BP\mathbb{R}$.
- (Dugger) $G = C_2$, $E = k\mathbb{R}$.

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Motivation II

Motivic

- (Voevodsky; Levine; Hopkins–Morel) For $E \in \mathrm{SH}(k)$, effective slice spectral sequence

$$E_1^{m,q,n} = \pi^k_{m,n} s_q E \Rightarrow \pi^k_{m,n} E.$$

- (Röndigs–Spitzweck–Østvær) $\pi^k_{**}\mathbb{S}$ and variants of K-theory.

Examples

- (Yagita) $E = MGL$ and $E = BPGL$.
- (Weibel, Rognes, Kahn, Röndigs–Østvær, ...) $E = KGL$.

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Main Theorems I

From now on: Implicitly 2-complete.

Theorem A (Culver–Kong–Q.)

For $G = C_2$:

- Completely understand HHR SSS for $E = k\mathbb{R}$,
- Understand part of HHR SSS for $E = BP\mathbb{R}$ and $E = k\mathbb{R}(n)$.

Theorem B (Culver–Kong–Q.)

- ($k = \mathbb{R},\ \mathbb{C}$) Completely understand ESSS for $E = kgl,\ BPGL,\ k(n)$.
- ($k = \mathbb{F}_q$) Completely understand ESSS for $E = kgl,\ k(n)$.

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Main Theorems II

From now on: $\mathrm{char}(k) = 0$.

Theorem C (Culver–Kong–Q.)

Let $E \in \mathrm{SH}(k)$ be a motivic spectrum which is algebraically sliceable over k. There is a square of spectral sequences of the form

$$
\begin{array}{ccccc}
 & & \bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A_k^\vee}(\widehat{s}_q E) & & \\
 & \overset{\widehat{aESSS}}{\swarrow} & & \overset{\bigoplus_{q\geq 0} mASS}{\searrow} & \\
\mathrm{Ext}^{***}_{A_k^\vee}(E) & & & & \bigoplus_{q\geq 0} \pi^k_{**}\widehat{s}_q E \\
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Main Theorems III

Theorem D (Culver–Kong–Q.)

The following motivic spectra are algebraically sliceable:

- kgl
- $BPGL\langle n\rangle$
- $k(n)$
- kq

Non-examples

- $BPGL$
- $\mathbb{S}$

Main Theorems III

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- $BPGL\langle n\rangle$
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Non-examples

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Plan for talk

1. Theorem C (Comparison square)
2. Theorem D (algebraic sliceability) for *kgl*
3. Theorem B (ESSS) for *kgl*
4. Theorem A (HHR SSS) for $k\mathbb{R}$

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Effective slice spectral sequence

Effective slice tower

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Effective slice spectral sequence

$$\widehat{\mathsf{ESSS}}E^1_{m,q,n}(E) = \pi^k_{m,n}\widehat{s}_qE \Rightarrow \pi^k_{m,n}E.$$

Example

$$E = kgl = f_0 KGL.$$

$$f_q kgl \simeq \Sigma^{2q,q} kgl.$$

$$s_q kgl \simeq \begin{cases} \Sigma^{2q,q} H\mathbb{Z}, & \text{if } q \geq 0, \\ 0 & \text{if } q < 0. \end{cases}$$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Sigma^{4,2} kgl & \longrightarrow & \Sigma^{2,1} kgl & \longrightarrow & kgl \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Sigma^{4,2} H\mathbb{Z} & & \Sigma^{2,1} H\mathbb{Z} & & H\mathbb{Z}. \end{array}$$

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Motivic Adams spectral sequence

Canonical Adams resolution

$$\begin{array}{ccccccc} E & \longleftarrow & \Sigma^{-1,0}\overline{H} \wedge E & \longleftarrow & \Sigma^{-2,0}\overline{H}^{\wedge 2} \wedge E & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ H \wedge E & & \Sigma^{-1,0} H \wedge \overline{H} \wedge E & & \Sigma^{-2,0} H \wedge \overline{H}^{\wedge 2} \wedge E & & \end{array}$$

Motivic Adams spectral sequence

$$^{\mathrm{mASS}}E_2^{s,f,w} = \mathrm{Ext}_{A_k^\vee}^{s,f,w}(H_{**}, H_{**}(E)) \Rightarrow \pi_{s,w}^k(E).$$

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Comparison almost-square

Let $E \in \mathrm{SH}(k)$ be a motivic spectrum. There are spectral sequences of the form

$$\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A_k^\vee}(\widehat{s}_q E) \overset{\bigoplus_{q\geq 0} mASS}{\Longrightarrow} \bigoplus_{q\geq 0} \pi^k_{**}\widehat{s}_q E$$

$$\mathrm{Ext}^{***}_{A_k^\vee}(E) \overset{mASS}{\Longrightarrow} \pi^k_{**}E \overset{\widehat{ESSS}}{\Longleftarrow} \bigoplus_{q\geq 0} \pi^k_{**}\widehat{s}_q E$$

$$\pi^k_{**}E.$$

Algebraic effective slice spectral sequence

Want:

$$E^1_{m,q,n}(E) = \bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A_k^\vee}(H_{**}, H_{**}(\widehat{s}_q E)) \Rightarrow \mathrm{Ext}^{***}_{A_k^\vee}(H_{**}, H_{**}(E))$$

Use:

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Problem:

$$\mathrm{Ext}^{***}_{A_k^\vee}(H_{**}, H_{**}(-)) = \mathrm{Ext}^{***}_{A_k^\vee}(H_{**}, -) \circ H_{**}(-).$$

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Algebraically sliceable motivic spectra

Homology long exact sequence

$H_{**}(-)$ of SES

$$\widehat{f}_{q+1}E \to \widehat{f}_q E \to \widehat{s}_q E.$$

yields LES's

$$\cdots \to H_{**}\widehat{f_{q+1}}E \to H_{**}\widehat{f}_q E \to H_{**}\widehat{s}_q E \to H_{*-1,*}\widehat{f}_{q+1}E \to \cdots.$$

Algebraically sliceable motivic spectra

$E \in \mathrm{SH}(k)$ is *algebraically sliceable over k* if the LES's above split into SES's

$$0 \to H_{**}\widehat{f}_q E \xrightarrow{p} H_{**}\widehat{s}_q E \xrightarrow{\iota} H_{*-1,*}(\widehat{f}_{q+1}E) \to 0$$

for each $q \geq 0$.

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Slice tower for *kgl*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & f_2 kgl & \longrightarrow & f_1 kgl & \longrightarrow & kgl \\
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Goal

Show $H_{**}(-)$ LES for

$$f_{q+1} kgl \to f_q kgl \to s_q kgl$$

splits into SES for all $q \geq 0$.

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Sketch of algebraic sliceability

$$f_{q+1}kgl \to f_q kgl \to s_q kgl.$$

$$\cdots \to H_{**}f_{q+1}kgl \xrightarrow{\beta} H_{**}f_q kgl \to H_{**}s_q kgl \to H_{*-1,*}f_{q+1}kgl \xrightarrow{\beta} \cdots,$$

$$H_{**}(f_{q+1}kgl \to f_q kgl) = H_{**}(\beta) = 0.$$

$$0 \to H_{**}f_q kgl \xrightarrow{p} H_{**}s_q kgl \xrightarrow{\iota} H_{*-1,*}(f_{q+1}kgl) \to 0.$$

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Outline

Theorem B for *kgl*

Goal

Completely understand the ESSS for *kgl* over $k = \mathbb{R}$

$$E^1 = \bigoplus_{q \geq 0} \pi^{\mathbb{R}}_{**} s_q kgl \Rightarrow \pi^{\mathbb{R}}_{**} kgl.$$

Comparison square for *kgl*

$$\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl)$$

$$\text{aESSS}: \bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl) \Rightarrow \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl)$$

$$\bigoplus_{q\geq 0} \text{mASS}: \bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl) \Rightarrow \bigoplus_{q\geq 0} \pi^{\mathbb{R}}_{**} s_q kgl$$

$$\text{mASS}: \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl) \Rightarrow \pi^{\mathbb{R}}_{**} kgl.$$

$$\text{ESSS}: \bigoplus_{q\geq 0} \pi^{\mathbb{R}}_{**} s_q kgl \Rightarrow \pi^{\mathbb{R}}_{**} kgl.$$

New goal

ESSS via $\oplus$mASS, mASS, and aESSS.

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mASS's collapse

Proposition (Hill)

The $\mathbb{R}$-motivic Adams spectral sequences

$$E_2 = \bigoplus_{q \geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl) \Rightarrow \bigoplus_{q \geq 0} \pi^{\mathbb{R}}_{**} s_q kgl,$$

$$E_2 = \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl) \Rightarrow \pi^{\mathbb{R}}_{**} kgl$$

both collapse at E_2.

Comparison square for *kgl*

$$\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl)$$

aESSS (↙) $\qquad$ $\bigoplus_{q\geq 0}$ mASS (↘)

$$\mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl) \qquad\qquad \bigoplus_{q\geq 0} \pi^{\mathbb{R}}_{**} s_q kgl$$

mASS (↘) $\qquad$ ESSS (↙)

$$\pi^{\mathbb{R}}_{**} kgl.$$

New goal

ESSS via aESSS

Comparison square for *kgl*

$$
\begin{array}{ccccc}
 & & \bigoplus_{q\geq 0} \operatorname{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl) & & \\
 & \overset{aESSS}{\swarrow} & & \overset{\bigoplus_{q\geq 0} mASS}{\searrow} & \\
\operatorname{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl) & & & & \bigoplus_{q\geq 0} \pi^{\mathbb{R}}_{**} s_q kgl \\
 & \underset{mASS}{\searrow} & & \underset{ESSS}{\swarrow} & \\
 & & \pi^{\mathbb{R}}_{**} kgl. & &
\end{array}
$$

New goal

ESSS via aESSS

Another square

$$\begin{array}{ccc}
\bigoplus_{q\geq 0} \operatorname{Ext}^{***}_{A^\vee_{\mathbb{C}}}(s_q kgl)[\rho] & \overset{aESSS}{\Longrightarrow} & \operatorname{Ext}^{***}_{A^\vee_{\mathbb{C}}}(kgl)[\rho] \\
\oplus\rho{-}BSS \Big\Downarrow & & \Big\Downarrow \rho{-}BSS \\
\bigoplus_{q\geq 0} \operatorname{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl) & \overset{aESSS}{\Longrightarrow} & \operatorname{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl)
\end{array}$$

New goal

Understand aESSS via ρ-BSS and aESSS.

Another square

$$\begin{array}{ccc}
\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(s_q kgl)[\rho] & \xrightarrow{aESSS} & \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(kgl)[\rho] \\
\oplus\rho{-}BSS \Downarrow & & \Downarrow \rho{-}BSS \\
\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl) & \xrightarrow{aESSS} & \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl)
\end{array}$$

New goal

Understand aESSS via ρ-BSS and aESSS.

ρ-Bockstein spectral sequence

- Classical:

$$H^{cl}_{**} = \mathbb{F}_2$$
$$A^{\vee}_{cl} = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

- $\mathbb{C}$-motivic:

$$H^{\mathbb{C}}_{**} = \mathbb{F}_2[\tau]$$
$$A^{\vee}_{\mathbb{C}} = A^{\vee}_{cl} \otimes_{\mathbb{F}_2} H^{\mathbb{C}}_{**}$$

- $\mathbb{R}$-motivic:

$$H^{\mathbb{R}}_{**} = \mathbb{F}_2[\tau, \rho]$$
$$\text{“}A^{\vee}_{\mathbb{R}} = A^{\vee}_{\mathbb{C}} \otimes_{H^{\mathbb{C}}_{**}} H^{\mathbb{R}}_{**}\text{”}$$

- (Hill) ρ-Bockstein spectral sequence

$$^{\rho-\mathrm{BSS}}E_1(E) = \mathrm{Ext}^{***}_{A^{\vee}_{\mathbb{C}}}(E)[\rho] \Rightarrow \mathrm{Ext}^{***}_{A^{\vee}_{\mathbb{R}}}(E).$$

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Another square

$$\begin{array}{ccc} \bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(s_q kgl)[\rho] & \overset{aESSS}{\Longrightarrow} & \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(kgl)[\rho] \\ \oplus\rho{-}BSS \Big\Downarrow & & \Big\Downarrow \rho{-}BSS \\ \bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(s_q kgl) & \overset{aESSS}{\Longrightarrow} & \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl) \end{array}$$

New goal

Understand aESSS via ρ-BSS and aESSS.

ρ-Bockstein spectral sequences

- $$^{\rho-\mathrm{BSS}}E_1(H\mathbb{Z}) = \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(H\mathbb{Z})[\rho] \Rightarrow \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(H\mathbb{Z}),$$
$$^{\rho-\mathrm{BSS}}E_1(kgl) = \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(kgl)[\rho] \Rightarrow \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(kgl).$$
- $$^{\rho-\mathrm{BSS}}E_1(H\mathbb{Z}) \cong \mathbb{F}_2[v_0, \tau, \rho],$$
$$^{\rho-\mathrm{BSS}}E_1(kgl) \cong \mathbb{F}_2[v_0, v_1, \tau, \rho],$$
- $H\mathbb{Z}$ and kgl:
$$d_1(\tau) = \rho v_0.$$
- kgl:
$$d_3(\tau^2) = \rho^3 v_1.$$

ρ-Bockstein spectral sequences

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  $$ {}^{\rho-\mathrm{BSS}}E_1(kgl) \cong \mathbb{F}_2[v_0, v_1, \tau, \rho], $$
- $H\mathbb{Z}$ and *kgl*:
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ρ-Bockstein spectral sequence for $H\mathbb{Z}$

$$^{\rho-\mathrm{BSS}}E_1(H\mathbb{Z}) \cong \mathbb{F}_2[v_0, \tau, \rho] \Rightarrow \mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(H\mathbb{Z}),$$

$$d_1(\tau) = \rho v_0.$$

Comes from

$$\eta_L(\tau) - \eta_R(\tau) = (\tau + \rho\tau_0) - \tau = \rho\tau_0,$$

$$[\rho\tau_0] = \rho v_0.$$

$$E_2 = E_\infty \cong \mathbb{F}_2[v_0, \tau^2, \rho]/(\rho v_0).$$

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$$^{\rho-\mathrm{BSS}}E_1(kgl) \cong \mathbb{F}_2[v_0, v_1, \tau, \rho] \Rightarrow \mathrm{Ext}^{***}_{A^\vee_\mathbb{R}}(kgl),$$

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$$^{\rho-\mathrm{BSS}}E_2(kgl) \cong \mathbb{F}_2[v_0, v_1, \tau^2, \rho]/(\rho v_0),$$

$$d_3(\tau^2) = \rho^3 v_1.$$

Comes from

$$\eta_L(\tau^2) - \eta_R(\tau^2) \equiv (\tau^2 + (\rho\tau_0)^2) - \tau^2 \equiv \rho^3\tau_1 \mod \rho^4,$$

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\begin{array}{ccc}
\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(s_q kgl)[\rho] & \overset{aESSS}{\Longrightarrow} & \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(kgl)[\rho] \\
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Summary

ρ-BSS: d_1- and d_3-differentials

$\oplus$ ρ-BSS: d_1-differentials

Remaining: aESSS and aESSS.

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aESSS over $\mathbb{C}$

- aESSS:

$$\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(s_q kgl)[\rho] \Rightarrow \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(kgl)[\rho]$$

- LHS:

$$\bigoplus_{q\geq 0} \mathrm{Ext}^{***}_{A^\vee_{\mathbb{C}}}(s_q kgl)[\rho] \cong \mathbb{F}_2[v_0, x, \tau, \rho]$$

- RHS:

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$$x \leftrightarrow v_1$$

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\end{array}$$

Summary

ρ-BSS: d_1- and d_3-differentials
$\oplus$ ρ-BSS: d_1-differentials
aESSS: collapses

Conclusion

aESSS: d_1-differentials accounting for d_3-differentials from ρ-BSS

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Theorem B for *kgl*

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\mathrm{Ext}^{***}_{A^\vee_{\mathbb{R}}}(E) & & & & \bigoplus_{q\geq 0} \pi^{\mathbb{R}}_{**} s_q E \\
 & \underset{mASS}{\searrow} & & \underset{ESSS}{\swarrow} & \\
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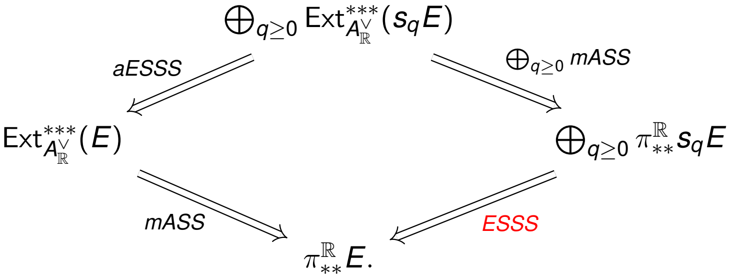
Theorem B for *kgl* (Culver–Kong–Q.)

- There is a 1-to-1 correspondence between d_3-differentials in the ρ-BSS and d_1-differentials in the ESSS for *kgl*.
- The ρ-BSS differentials determine the ESSS differentials, which are generated by

$$d_1(\tau^2) = \rho^3 v_1.$$

Theorem B for *kgl*



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Outline

Motivation and main theorems

Theorem C: Comparison square

Theorem D: *kgl* is algebraically sliceable

Theorem B: ESSS for *kgl*

Theorem A for $k\mathbb{R}$

Future work

Sketch of Theorem A for $k\mathbb{R}$

Theorems

- (CKQ) The d_1-differentials in the ESSS for *kgl* are generated by
$$d_1(\tau^2) = \rho^3 v_1.$$
- (Heard) In nice cases, d_1-differentials in the ESSS give rise to d_3-differentials in the HHR SSS via
$$\mathrm{Re} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(C_2).$$

Theorem A for $k\mathbb{R}$

- $$d_3(\tau^2) = \rho^3 \bar{v}_1$$
- $$d_3\left(\frac{\gamma}{\rho^3\tau^2}\right) = \frac{\gamma}{\tau^4}\bar{v}_1.$$

Sketch of Theorem A for $k\mathbb{R}$

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Theorem A for $k\mathbb{R}$

-

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-

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Theorems

- (CKQ) The d_1-differentials in the ESSS for *kgl* are generated by

$$d_1(\tau^2) = \rho^3 v_1.$$

- (Heard) In nice cases, d_1-differentials in the ESSS give rise to d_3-differentials in the HHR SSS via

$$\mathrm{Re} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(C_2).$$

Theorem A for $k\mathbb{R}$

- $$d_3(\tau^2) = \rho^3 \bar{v}_1$$

- $$d_3\left(\frac{\gamma}{\rho^3 \tau^2}\right) = \frac{\gamma}{\tau^4} \bar{v}_1.$$

Negative cone differentials

$$\frac{\gamma}{\rho^3\tau^2} \cdot \tau^2 = 0$$

$$0 = d_3\left(\frac{\gamma}{\rho^3\tau^2} \cdot \tau^2\right) = d_3\left(\frac{\gamma}{\rho^3\tau^2}\right) \cdot \tau^2 + \frac{\gamma}{\rho^3\tau^2} \cdot d_3(\tau^2)$$

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Outline

Future work

1. kq: mASS and ESSS
2. Exotic periodicity:
   2.1 (Gheorghe) $wBP\langle n\rangle$
   2.2 (Krause) $k(i,j)$
3. $kq_{**}(BG)$ and $kgl_{**}(BG)$
   - (Bruner–Greenlees) $ko_*(BG)$ and $ku_*(BG)$
4. $N_{C_2}^{C_{2^n}} BP\mathbb{R}\langle n\rangle$