Math 4530 Homework 1.

Reminder:
· Discussing with classmates is encouraged, but you must write up your solutions alone!
· Searching on the internet (or elsewhere) for solutions is *prohibited.* This is a discovery-based course where you will learn concepts by playing with examples and struggling with problems.

"The only way to learn mathematics is to do mathematics" - P. Halmos

1. Read the course syllabus carefully!

2. Tell me about yourself:
   1. Year (sophomore, senior, grad student, etc.)
   2. Major(s) or intended major(s) or interests
   3. What 3000 and 4000-level math courses have you taken, and which are you currently taking?
   4. Anything else you would like to share?

3. Let $X, \mathcal{O}$ be a topological space.
   (a) Show that a subset $A \subset X$ is open if and only if $\mathring{A} = A$.
   (b) Show that
   $$\overline{A} = \bigcap_{Y \text{ closed},\, Y \supseteq A} Y$$
   (c) Prove the following holds, or find a topological space that provides a counterexample: *For any collection of sets A_λ, the union of their closures is the closure of their union, i.e.*
   $$\bigcup_\lambda \overline{A_\lambda} = \overline{\bigcup_\lambda A_\lambda}$$
   (d) Do the problem above, but replace union with intersection.

4. Prove that the following "neighborhood axioms" give an equivalent definition of a topological space to that given in class, if you define open sets to be those that contain a neighborhood of each point in the set. [1]
   Definition. A topological space is a set X and a collection of subsets called "neighborhoods of points" satisfying the following.
   1. There corresponds to each point x at least one neighborhood of x, and each neighborhood of x contains the point x.
   2. If V contains a neighborhood of x, then V is itself a neighborhood of x.
   3. The intersection of any two neighborhoods of x is itself a neighborhood of x.
   4. If a point y lies in a neighborhood U_x of some point x, then there must exist a neighborhood U_y of y that is a subset of U_x.

5. In this question, you will show that any metric topology can be defined by a metric where the distance between any two points is at most 1. In other words, topologists cannot see how big something is!
   Suppose that X is a set and d a metric on X.

[1] Historical remark: Hausdorff made this definition in 1919. He also proposed the following additional axiom:
5. For two different points x and y, there are two corresponding neighborhoods U_x and U_y with no points in common.
You can show that this is not equivalent: there are topological spaces (according to our definition) which do not satisfy this extra axiom. Can you think of some? (you do not need to hand this in)

(a) Show that $d'(x,y) = \min\{d(x,y), 1\}$ satisfies the axioms of a metric
(b) Show that $d''(x,y) = \frac{d(x,y)}{1+d(x,y)}$ satisfies the axioms of a metric.
(c) It is a fact that both of these define the same topology on X as d. Prove this fact for the metric d'' defined in part b of this question. (You can check d' for extra practice but you don't need to hand that in).

6. Let X be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. Define distances:
$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$$
and define
$$d'(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$
These both satisfy the axioms needed for a metric space – pick *one of them* and show that all the axioms hold. For the one that you chose, state in words what the distance is measuring.

7. Define a "neighborhood" of a point $(x,y) \in \mathbb{R}^2$ to be any set of the form $I \times \{y\}$ where I is an open interval (in the usual sense of the word open for intervals of $\mathbb{R}$) containing x. Let $\mathcal{O}$ be the set of arbitrary unions of "neighborhoods" together with the empty set. Does this define a topology on $\mathbb{R}^2$? If not, why not? If so, are the x and the y axes open sets or closed sets or neither?

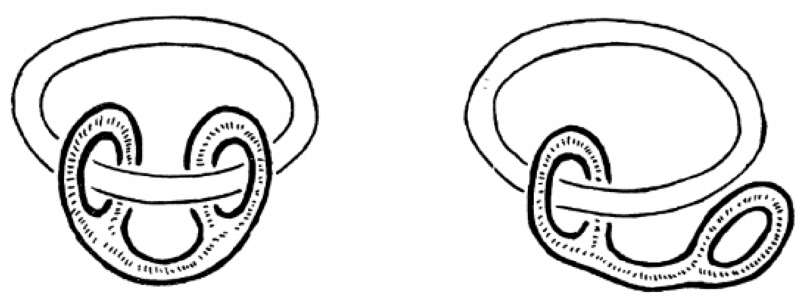
8. Let $A = \{0\} \cup [1,2] \cup \{p \mid p \in \mathbb{Q}, p > 3\}$, a subset of $\mathbb{R}$ with the usual topology. How many different subsets of $\mathbb{R}$ can you attain by starting with A and taking closures, complements, and interiors (perhaps iteratively, so you're allowed to take the closure of the complement, for example).

   (b) If you start with a different set than A, can you get more possibilities?

   Challenge (1 bonus point): What is the maximum number? Is there even a maximum?

9. Do the following problem about continuous deformations (you need an explanation, with pictures, not a rigorous proof). (see also the back of the warm-up from 8/29)

   ***Problem* 1.2. A pretzel has two holes that "hold" a doughnut (see Figure 1.2 (a)). Show that the pretzel can be deformed in such a way that one of its "handles" will unlink itself from the doughnut (Figure 1.2 (b)).**



(a) (b)

FIGURE 1.2

10. **Challenge (not for credit)**: Solve the n-pancake problem in $\mathbb{R}^n$: if you have n solid objects in $\mathbb{R}^n$, then there exists a straight line cut (an $n-1$ dimensional plane) that simultaneously divides the volume of each of them in half.