Math 4530 Homework 10: Lifting paths and loops.

We are usually convinced more easily by reasons we have found ourselves than by those which occurred to others. - Blaise Pascal

Recommended reading:
Jänich, chapter IX sections 2-5. (we'll continue with section 5 next week). Alternative/additional: Munkres chapter 51

Problems:

For the first two questions, we again consider S^1 as $\mathbb{R}/\sim$ where $x \sim y$ if $x = y + k$ for some $k \in \mathbb{Z}$. The function Δ is that defined on the previous homework takes integer values.

1. Use the homotopy lifting theorem and/or the monodromy lemma to give a new, short and complete proof that the function Δ is
   a) well-defined (meaning it doesn't depend on which lift you choose) and
   b) assigns the same value to functions that are homotopic.

2. Let f and g be continuous maps from $S^1 \to S^1$. Prove that $\Delta(f \circ g) = \Delta(f)\Delta(g)$.

3. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ given by $(x, y) \sim (p, q)$ if $x - p \in \mathbb{Z}$ and $y - q \in \mathbb{Z}$. Prove that the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\sim$ is a covering map.

4. Let $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Give an example of a loop α based at $\pi((0, 0))$ in $\mathbb{R}^2/\sim$ so that the lift of α based at $(0, 0)$ has its other endpoint at (m, n).

   For the next question, we say that two paths (or loops) f and g from $[0, 1] \to X$ are said to be *homotopic rel endpoints* if $f(0) = g(0)$, and $f(1) = g(1)$, and there is a homotopy f_t with $f_0 = f$ $f_1 = g$, and such that $f_t(0) = f(0)$ and $f_t(1) = f(1)$ for all t.

5. Let X be a connected topological space, and let $\pi : Y \to X$ be a cover. Let $x_0 \in X$. Let $\alpha : [0, 1] \to X$ and $\beta : [0, 1] \to X$ be two loops based at x_0. The *concatenation* $\alpha\beta$ is a loop defined by
   $\alpha\beta(t) = \alpha(2t)$ if $t \in [0, 1/2]$, and
   $\alpha\beta(t) = \beta(2t - 1)$ if $t \in [1/2, 1]$.
   Show that if α' is homotopic to α rel endpoints and β' is homotopic to β rel endpoints, then $\alpha\beta$ is homotopic to $\alpha'\beta'$ rel endpoints.

6. Let $\alpha : [0, 1] \to S^1$ be the map $x \to [x]$ (considering $S^1 = \mathbb{R}/\sim$ as usual, so this is the map that just glues the two ends of the interval together, wrapping it once around the circle). Let $\beta : [0, 1] \to S^1$ be the map $x \to [1 - x]$.
   Check that $\alpha\beta(0) = \alpha\beta(1)$, so we can think of $\alpha\beta$ as a function from S^1 to S^1. Then compute the value of $\Delta(\alpha\beta)$.