Math 4530 Homework 11: The fundamental group.

In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics. - Hermann Weyl

Recommended reading:
Jänich, chapter IX sections 2-5. Alternative/additional: Munkres chapters 52-54

Problems:

1. Let X be a topological space. Let $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$ be two paths, with $\alpha(1) = \beta(0)$. The *concatenation* $\alpha\beta$ is the path defined by
$\alpha\beta(t) = \alpha(2t)$ if $t \in [0, 1/2]$, and
$\alpha\beta(t) = \beta(2t - 1)$ if $t \in [1/2, 1]$.
Suppose $\beta(t) = \alpha(1 - t)$. Show that $\alpha\beta$ is homotopic rel endpoints to a constant map.

   Now use this to conclude that inverses exist in $\pi_1(X, x_0)$

2. Show that concatenation of loops is associative, i.e. if α, β and γ are loops based at x_0 in a topological space X, then $\alpha(\beta\gamma)$ is homotopic as a based loop to $(\alpha\beta)\gamma$.
(Remember that a *based loop* is a map from $[0, 1]$ to X that sends 0 and 1 to x_0, and a homotopy h_t of based loops requires $h_t(0) = h_t(1) = x_0$ for all t.)

3. In the previous homework, you showed that the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\sim$ is a covering map, where $(x, y) \sim (p, q)$ if $x - p \in \mathbb{Z}$ and $y - q \in \mathbb{Z}$.
Let x_0 be the equivalence class of $(0, 0)$ in the quotient space $\mathbb{R}^2/\sim$. If $\gamma : [0, 1] \to \mathbb{R}^2/\sim$ is a loop based at x_0, let $\hat{\gamma}$ denote the unique lift of γ to a path $[0, 1] \to \mathbb{R}^2$ with $\hat{\gamma}(0) = (0, 0)$.

   Define a map $\Phi : \pi_1(\mathbb{R}^2/\sim, x_0) \to \mathbb{Z} \times \mathbb{Z}$ by $\Phi([\gamma]) = \hat{\gamma}(1)$. Show that this is
a) Well-defined (it depends only on the homotopy class of γ as a based loop)
b) Surjective
c) Respects composition: if α and β are two loops, then $\Phi([\alpha\beta]) = \Phi([\alpha]) + \Phi([\beta])$.
d) (Not required to hand in) Check also that Φ is injective.

   Thus, we can conclude from the above that $\pi_1(\mathbb{R}^2/\sim, x_0)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

4. Let S^2 be the unit sphere (with the subset topology from $\mathbb{R}^3$) and let $x_0 \in S^2$ be a point. Prove (using the fact below) that $\pi_1(S^2, x_0) = \{id\}$. Conclude that S^2 is not homeomorphic to T^2.

   FACT (you may use this without proof): If $p \neq x_0$ and $\alpha : [0, 1] \to S^2$ is a continuous loop based at x_0, then α is homotopic as a based loop to a loop β whose image does not contain p.
BONUS: prove the fact.

5. Recall that, if $f : X \to Y$ is continuous, then it induces a map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ via $f_*([\alpha]) = [f \circ \alpha]$.
Check that $f_*(\alpha\beta) = f_*(\alpha)f_*(\beta)$, so this map is a group homomorphism.

6. Suppose that Y is a connected topological space, and $\pi : Y \to \mathbb{R}^n$ is a covering map. Show that π is a homeomorphism.