Math 4530 Homework 12: Final HW set!

To many, mathematics is a collection of theorems. For me, mathematics is a collection of examples; a theorem is a statement about a collection of examples and the purpose of proving theorems is to classify and explain the examples... - John B. Conway.

Problems:

1. Recall that a subset A of a topological space X is a *deformation retract* of X if there is a homotopy $H : X \times [0,1] \to X$ such that $H(x,0) = x$, $H(x,1) \in A$, and $H(a,t) = a$ holds for all $x \in X$, $a \in A$ and $t \in [0,1]$.

   a) Find a circle inside the Mobius band such that the loop is a deformation retract of the Mobius band.
   b) The edge of the Mobius band is also a circle. Show that this is not a deformation retract of the Mobius band. (possible approach: look at the fundamental group!)

2. Suppose X is a topological space and $B \subset A \subset X$. If B is a deformation retract of A and A is a deformation retract of X, show that B is a deformation retract of X.

3. Show that the two figures below (an eight and a "theta") have isomorphic funadmental group. Possible strategy: define a bigger space X and use deformation retracts...
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4. Let $X \subset \mathbb{R}^2$ be the topological space consisting of the union of the lines $[0,1]\times\{0\}$, $\{0\}\times[0,1]$ and $\{1/n\} \times [0,1]$ for all $n \in \mathbb{N}$. Give this the subset topology from $\mathbb{R}^2$.

   a) Show that X is not locally path connected
   b) Show that the identity map $X \to X$ is homotopic to the constant map $X \to X$ whose image is the single point $(0,0)$.
   c) Using part b, show that the identity map $X \to X$ is homotopic to the constant map whose image is the single point $(0,1)$
   d) Show that the point $(0,1) \in X$ is *not* a deformation retract of X. Hint: suppose it was a deformation retract. Let $z_n = (1/n, 1)$ and try to understand $H(t, z_n)$
   e) Why does part c) not contradict part d) ?

5. Let X be a topological space and $A \subset X$. We say A is a *retraction of* X if there is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. We call r a retraction.

   a) Prove that if $r : X \to A$ is a retraction of X, and $x_0 \in A$, then the map $r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective.
   b) Give an example of a space X and a subset $A \subset X$ that is a retraction but not a deformation retract.

6. For $k \in \mathbb{N}$, let $p_k : S^1 \to S^1$ be the "multiply the angle by k map". If you think of S^1 as $\mathbb{R}/\!\sim$ then this is the map $p_k([x]) = [kx]$. We know that $p_k : S^1 \to S^1$ is a cover.

   For which continuous maps $f : S^1 \to S^1$ is there a lift to the cover $p_k : S^1 \to S^1$? Give a complete list (which will probably depend on k) and justify your answer.
   Hint: Think about $\Delta(f)$. Compute $(p_k)_*$ and f_* and use the lifting criterion.

7. Compute the fundamental group of the following spaces. (You may take any basepoint you like!)

   a) $\mathbb{R}^3$ with the x-axis removed
   b) The space X from problem 4.

8. In this question, B^{n+1} denotes the unit radius ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ in $\mathbb{R}^{n+1}$, and S^n the unit radius sphere. Assume that the following statement holds: There is no retraction $r : B^{n+1} \to S^n$ for each n.

   a) Prove that the identity map $S^n \to S^n$ is not homotopic to a constant map.
   b) Prove that the inclusion map $i : S^n \to \mathbb{R}^{n+1} - \{\vec{0}\}$ is not homotopic to a constant map.
   c) Explain why every continuous map $f : B^{n+1} \to B^{n+1}$ must have a fixed point, i.e. $f(x) = x$ or some $x \in B^{n+1}$.