Math 4530 Homework 2: products, subspaces, continuity, connectedness

"I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives." - Charles Hermite, in a letter to Thomas Stieltjes

1. Recall the *frontier* of a subset A of a topological space X is the set of points that are neither interior points or A nor interior points of $X \setminus A$.[1] Prove just using the definitions that the frontier of any set is a closed set.

2. For which numbers $a < b \in \mathbb{R}$ is the interval $[a, b) \cap \mathbb{Q}$ open in the subspace topology on $\mathbb{Q}$? For which is it closed? (justify this briefly, don't just list the numbers a and b that work.)

3. Look at the topology on $\mathbb{R}^2$ defined in problem 7 from HW1. Describe the subset topology on the x-axis and on the y-axis.

4. Suppose that $X, \mathcal{O}_X$ and $Y, \mathcal{O}_Y$ are topological spaces and $\mathcal{O}_{X\times Y}$ the product topology on $X \times Y$. Let $A \subset X$ and $B \subset Y$, and give them the subspace topologies. Show that the product topology on $A \times B$ is the same as viewing $A \times B$ as a subset of the topological space $X \times Y$, $\mathcal{O}_{X\times Y}$ and giving it the subspace topology.

5. In class we talked about products of two spaces, but one can also take products of arbitrary collections of spaces. If X_n are topological spaces, indexed by $n \in \mathbb{N}$, then the *product topology* is defined to be that generated by the basis $\prod_{n\in\mathbb{N}} U_n$ where $U_n \subset X_n$ is open and $U_n = X_n$ for all but finitely many n.
   (a) An important example is where $X_n = \mathbb{R}$ for all n, then we can think of $\prod_{n\in\mathbb{N}} X_n$ as the set of all sequences of real numbers $(a_1, a_2, \ldots)$. For $k \in \mathbb{N}$, $\epsilon > 0$, and $x_n \in X_n$ define the set
   $$O_{\epsilon,x_1,\ldots x_k} := \{(a_1, a_2, \ldots) : |a_i - x_i| < \epsilon \text{ for all } i \leq k\}$$
   Is the set $\{O_{\epsilon,x_1,\ldots x_k} : x_i \in X, \epsilon > 0\}$ a basis for the product topology on $\prod_{n\in\mathbb{N}} X_n$ in this case?
   (b) Show that, in general, the product of finitely many discrete topological spaces is discrete, but the product topology on $\prod_{n\in\mathbb{N}} Y_n$ where $Y_n = \{0, 1\}$ with the discrete topology, is not discrete.
   (c) Is the same true if you do the question above, but instead of the product topology you take the topology on $\prod_{n\in\mathbb{N}} X_n$ generated by the basis $\{\prod_{n\in\mathbb{N}} U_n : U_n \text{ open in } X_n\}$?

6. Let X and Y be topological spaces, and for a point $x \in X$ let $Y_x = \{(x, y) : y \in Y\}$. Show that for any $x \in X$, the set Y_x is homeomorphic to Y.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that f gives a homeomorphism between $\mathbb{R}$ and its graph (a subspace of $\mathbb{R}^2$).

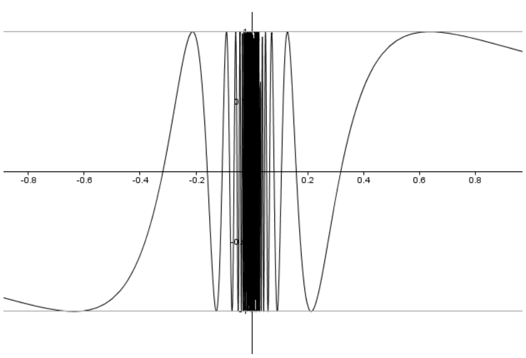
[1] I will be careful to use the word "frontier" for this, not "boundary" I asked several mathematicians "what is the boundary of the open interval $(0, 1)$" and 4 out of 6 responded "the points 0 and 1" (others asked for clarification), but to the question "does the open interval $(0, 1)$ have boundary?" 5 out of 5 other mathematicians I asked said "no". For us, $(0, 1)$ does have frontier in $\mathbb{R}$ and it's the points 0 and 1. We'll write ∂A for frontier of A.

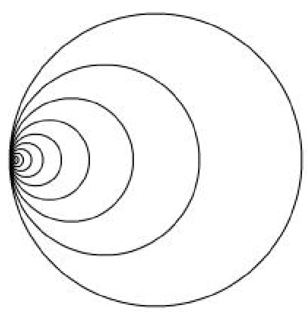
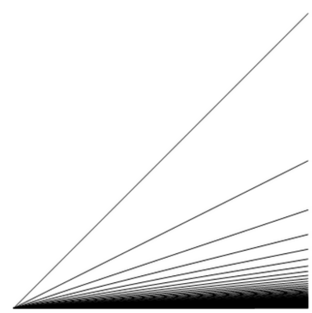
8. Let X be a topological space, and $A \subset X$. Suppose you have some topology $\mathcal{O}$ on A (not necessarily the subspace topology). Prove that, if the inclusion map $A \to X$ is a continuous map, then every open set from the subspace topology is in $\mathcal{O}$. (In other words, the subspace topology is the "smallest" topology that makes inclusions continuous.)

9. Let X be a topological space, and $A \subset X$ (with the subset topology). Prove or give a counterexample to the following statements:
   (a) If A is open and connected and $f : X \to \mathbb{R}$ a continuous map, then the image of A is an open interval.
   (b) If A is connected then $\overline{A}$ is connected
   (c) If A is path connected then $\overline{A}$ is path connected
   (d) If A is connected then the frontier of A is connected
   (e) If A is connected then $\overline{A}$ is path connected
   (f) If the frontier of A is connected then A is connected

10. Show that the product (with the product topology) of two connected topological spaces is connected.

11. For each of the following subspaces of $\mathbb{R}^2$ (with the subset topology coming from the usual topology on $\mathbb{R}^2$), determine if the space is connected and/or path connected. Justify your answer.
   (a) The graph of the function $y = \sin(\frac{1}{x})$ for $x \in [-1, 0) \cup (0, 1]$
   (b) The graph of the function $y = \sin(\frac{1}{x})$ for $x \in [-1, 0) \cup (0, 1]$, union the origin.
   (c) Let A_n denote a circle of radius $1/n$ and center at point $(1/n, 0)$ in $\mathbb{R}^2$. Answer the question for the subspace $\bigcup_{n \in \mathbb{N}} A_n$
   (d) Let B_n be the graph of the function $y = x/n$ where $x \in [0, 1]$. Answer the question for the subspace $\bigcup_{n \in \mathbb{N}} B_n$



12. **Challenge (not to hand in)**: Is the space from question 10 (b) homeomorphic to $\mathbb{R}$? Prove or disprove!

Extra references.
Some students have asked me for suggestions of other books to look at. Allen Hatcher (professor emeritus here and world-class topologist) has some good suggestions in the "point-set topology" section of this list: `https://pi.math.cornell.edu/~hatcher/Other/topologybooks.pdf`
I'll put the pdf file of his list on the course website as well.