Math 4530 Homework 3: homeomorphisms, convergence, compactness...

...I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. - A. Wiles

1. Redo question 11(b) from the previous homework, showing whether the space is connected and/or path connected.

   If you thought you wrote a perfect answer last time, you can say so and we'll just take the previous answer and use that. Otherwise, this is a chance to improve your proof or your exposition given what you have learned.

2. Let $(a, b) \subset \mathbb{R}$ be an open interval. Prove that $\mathbb{R}$ is homeomorphic to (a, b).

3. Prove that the following spaces are pairwise non-homeomorphic.
   (a) The unit circle in $\mathbb{R}^2$ (with the subset topology)
   (b) The set $\mathsf{X} = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1, \; x = y \text{ or } x = -y\}$
   (with the subset topology from $\mathbb{R}^2$)
   (c) The segment $[0, 1] \subset \mathbb{R}$. (with the subset topology from $\mathbb{R}$)
   Hint: can a space be disconnected after removing a point?

4. Say that a topological space X is *reasonable* if for every pair of points x, y where $x \neq y$, there is an open set containing x but not y, and an open set containing y but not x.
   (a) Show that X is reasonable if and only if, for every point $x \in X$ the set $\{x\}$ is closed.
   (b) If X is reasonable, and $Y \subset X$ has the subset topology, is Y reasonable? Prove or give a counterexample.

5. The definition of "reasonable" above sounds very close to our definition of "Hausdorff" from class. Here we will see they are not the same.
   Let $X = \mathbb{N}$. Let $\mathcal{O} = \{A \subset X : X - A \text{ is finite or empty }\} \cup \{\emptyset\}$.
   (a) Check that $\mathcal{O}$ is a topology on X.
   (b) Show that $\mathcal{O}$ is reasonable but not Hausdorff.

6. Let X_n be topological spaces. Recall the *product topology* on $\prod_{n \in \mathbb{N}} X_n$ was defined last time to be generated by the basis

$$\left\{ \prod_{n \in \mathbb{N}} U_n : U_n \subset X_n \text{ open, and } U_n = X_n \text{ for all but finitely many } n \right\}$$

   (a) Suppose that $X_n = \mathbb{R}$ with the standard topology. In this case, we sometimes write $\mathbb{R}^\infty$ for the set $\prod_{n \in \mathbb{N}} X_n$. Consider the following sequence of points in $\mathbb{R}^\infty$

$$(1, 0, 0, 0...), \; (0, 1, 0, 0...), \; (0, 0, 1, 0, ...) \, ...$$

   (the jth term in the sequence has a 1 in the jth place.) Does this sequence converge to something in $\mathbb{R}^\infty$ with the product topology? If so, what point(s) does it converge to?

   (b) Does the sequence

$$(1, 1, 1, ...), \; (1/2, 1/2, 1/2, ...), \; (1/3, 1/3, 1/3, ...), \; ...$$

   converge to something? What point(s) does it converge to?

(c) Suppose we instead wanted to use the topology on $\prod_{n\in\mathbb{N}} X_n$ that was generated by the basis

$$\left\{ \prod_{n\in\mathbb{N}} U_n : U_n \subset X_n \text{ open} \right\}$$

Do the examples from above converge in this *different* topology on the set $\mathbb{R}^\infty$? If so, what point(s) do they converge to?

7. Show that the ball $B_1(0) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is not compact by giving an open cover of $B_1(0)$ that has no finite subcover. (Also, you need to prove that your cover has no finite subcover!)

8. Show (again just using the definition of compact in terms of open covers) that the set $\{0\} \cup \bigcup_{n\in\mathbb{N}}\{1/n\}$, with the subset topology from $\mathbb{R}$, is compact, but the set $\bigcup_{n\in\mathbb{N}}\{1/n\}$ is not compact.

9. Show that $\mathbb{N}$ with the topology from question 5 is compact by showing that every open cover has a finite subcover. (This is a pretty weird example of a compact space.)

I will post some reading on compactness on the course website this weekend so that you have more examples – Jänich does not say very much about the topic and we will spend more time on it. You might wish to start the reading before Tuesday's class.