Math 4530 Homework 4: compactness

each senate district shall contain as nearly as may be an equal number of inhabitants [...] *and be in as compact form as practicable* - The NY state constitution

1. (a) Let X be a Hausdorff space, and $A \subset X$ and $B \subset X$ be two **compact** sets such that $A \cap B = \emptyset$. Prove that there exists open sets U and V, with $U \supset A$ and $V \supset B$ and $U \cap V = \emptyset$.

   (b) Give an example to show that (a) is not necessarily true if A is not assumed to be compact.

   (c) Give an example to show that (a) is not necessarily true if X is not Hausdorff.

2. (a) Suppose that X and Y are compact topological spaces. Show that the space $X \times Y$ with the product topology is compact.

   (b) Conversely, suppose that a product $X \times Y$ of topological spaces is compact, and prove that X and Y must also be compact.

3. Show that $\mathbb{R}^2$ is not homeomorphic to $[0,1] \times [0,1]$, a subset of $\mathbb{R}^2$. You may use any results from class or proved previously on homework.

4. **YOUR CHOICE:** Either hand in full solutions to 3 of the practice problems from the next page, *or* do the challenge question below.

5. **Challenge question** (optional, see question 3).

   (a) Prove that, if X, d is a sequentially compact metric space, then for every $r > 0$, the open cover $\{B_r(x) : x \in X\}$ has a finite subcover. (Possible hint: prove the contrapositive)

   (b) Using this, show that X, d is compact.

   (c) In class we proved the converse of what you just showed: every compact metric space is sequentially compact. Prove that this works if you replace "metric space" with "topological space $X, \mathcal{O}$ with a countable basis $\mathcal{B}$ for $\mathcal{O}$."

** See next page for info on the quiz **

Quiz info and practice problems

The quiz covers all material up until the definitions of compactness and sequential compactness. This is covered in Jänich chapter 1, except for sequential compactness, which we defined in class

The purpose of the quiz is to encourage you to take stock of what we've learned so far. (By the time we arrive at the prelim, that will be an overwhelming amount, hence the quiz now)

Suggestions for study

• Make a list of all the important definitions we have learned so far, and give an example to illustrate each.

• Many of the assertions in Jänich are left as "exercises" or "notes" or otherwise suggestions for the reader to fill in for practice. Take his suggestions and prove these! It's great practice, and one of these will almost certainly either be on the prelim or the quiz.

Problems for extra practice

1. Suppose that $f : X \to \mathbb{R}$ is a continuous function, and $A \subset X$ is open. If $f(A)$ is a single point, is $f(\overline{A})$ necessarily also a single point? Prove or give a counterexample.
2. Suppose X is a finite set. Then any metric topology on X is the discrete topology.
3. Let $\mathcal{O}$ be a topology on $\mathbb{R}$ with basis consisting of all sets of the form (a, ∞), $a \in \mathbb{R}$. Is this the same or different than the usual topology? Does it make a difference if you say subbasis instead of basis?
4. Prove (a fact that we keep using) that the metric topology from the usual metric on $\mathbb{R}^2$ is equivalent to the product topology from $\mathbb{R} \times \mathbb{R}$. (Where $\mathbb{R}$ is given the usual topology).
5. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. (with the usual topologies on each)
6. Is $\mathbb{R}$ homeomorphic to $[0, 1)$? (Here $[0, 1)$ is assumed to have the subset topology from $\mathbb{R}$). Prove or disprove.
7. Suppose that $A \subset X$, and that x_n is a sequence of points, where $x_n \in A$ for all n. If x_n converges to some point $x \in X$, prove that $x \in \overline{A}$.