Math 4530 Homework 5: spaces of functions

0. (Not to hand in). Review your previous homework sets and the solutions and comments.

The purpose of the rest of this homework is for you to practice writing out in detail some of what we covered in class.

1. Let X and Y be topological spaces. We say that a sequence of functions $f_n : X \to Y$ *pointwise converges* to a function f if, for every $x \in X$, the sequence $f_n(x)$ converges to $f(x)$ in Y.

   (a) Let $\mathcal{O}$ be the topology on the set of all functions $X \to Y$ defined by the subbasis $\mathcal{S} = \{S_{x,U} : x \in X, U \subset Y \text{ open }\}$, where $S_{x,U}$ is defined to be the set $S_{x,U} = \{f : X \to Y : f(x) \in U\}$.

   Show that a sequence of functions f_n pointwise converges to f if and only if f_n converges to f (in our definition of convergence for topological spaces) in the set of all functions $X \to Y$ with the topology $\mathcal{O}$

   (b) Why can we not say that the set $\mathcal{S} = \{S_{x,U} : x \in X, U \subset X \text{ open }\}$ is a basis for a topology?

   (c) Consider the functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Show that the sequence f_n pointwise converges a function that is not continuous.

2. (Extra practice, not to be graded).
   Recall that a sequence of functions $f_n : [0,1] \to \mathbb{R}$ *uniformly converges to* f means that $\sup\{|f(x) - f_n(x)| : x \in [0,1]\} \to 0$ as $n \to \infty$.
   In contrast to the question above, prove that if $f_n : [0,1] \to \mathbb{R}$ are all continuous functions, and f_n uniformly converges to a function f, then f is also continuous.

   This is one of the reasons that we are interested in different topologies on sets of functions!

   Comment: I decided to cover this material on functions instead of Chapter II of Jänich, since it requires less background. Both serve as interesting examples where topology meets other areas of math (and spaces of functions appear as one of the examples in Chapter II). The point is that topology is a powerful tool for many areas of mathematics. Topological spaces are not all doughnuts and mobius strips! Spaces of functions (and vector spaces, and topological groups, and...) are some of the most important examples.