Math 4530 Homework 6: quotients

A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking, and frustration. And proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks - G.-C. Rota

Reading: Chapter III, sections 1, 2, 3, 6 and 7 in Jänich.

Problems:

1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove the following universal property of the product space:
   There is a unique (up to homeomorphism) space $P, \mathcal{O}_P$ with the property that there exist continuous maps $\pi_X : P \to X$ and $\pi_Y : P \to Y$ such that, for any topological space Z and any continuous maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, there is a unique continuous map $h : Z \to P$ such that $f_1 = \pi_X \circ h$ and $f_2 = \pi_Y \circ h$.

2. a) Suppose that X is a path connected topological space and $\sim$ an equivalence relation on X. Show that $X/\sim$ is path connected.
   b) If X is not necessarily path-connected, show that $X/\sim$ has at most as many path-components as X. Give an example to show that $X/\sim$ may have fewer path-components than X does.

3. Suppose that X, d is a metric space, with the metric topology, and A is a closed subset of X. Show that the quotient space X/A (collapsing A to a point) is a Hausdorff topological space. Why does this fail if A is not closed?

4. Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x - y \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation, and that $\mathbb{R}/\sim$ is homeomorphic to the unit circle (the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as a subset of $\mathbb{R}^2$)

5. Define $\sim$ on $\mathbb{R}^2$ by $p \sim q$ if *both* of the coordinates of the vector $p - q$ are integers.
   (a) (Not to hand in) check that $\sim$ is an equivalence relation
   (b) Show that $\mathbb{R}^2/\sim$ is homeomorphic to the product of two copies of the unit circle. *possible hint: use the previous question*
   (c) Show that $\mathbb{R}^2/\sim$ is homeomorphic to the surface of revolution
   $$\{((2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi)) \in \mathbb{R}^3 : \theta \in [0, 2\pi), \phi \in [0, 2\pi)\}$$
   with the subset topology from $\mathbb{R}^3$. In this question, you may use (without proof) basic facts from calculus/other math classes about continuity of functions of real variables.

6. Define a relation $\sim$ on $\mathbb{R}^2$ by saying that $(x_1, y_1) \sim (x_2, y_2)$ if
   • $|x_1| < 1$, $|x_2| < 1$, and there exists $c \in \mathbb{R}$ such that both points lie on the graph of the function $y = \log(1 - x^2) + c$, or
   • $|x_1| \geq 1$ and $x_1 = x_2$.
   (a) Show that this is an equivalence relation
   (b) Draw a picture of the equivalence classes in the plane
   (c) Assume that $\mathbb{R}^2$ has the usual topology. Show that $\mathbb{R}^2/\sim$ is not Hausdorff.
   (d) **Bonus.** What is $\mathbb{R}^2/\sim$, as a topological space? Can you give a simple description of this (e.g. as some lines attached together...)

7. Let $\mathrm{GL}_2(\mathbb{R})$ denote the set of invertible 2×2 matrices. Define a relation $\sim$ on $\mathrm{GL}_2(\mathbb{R})$ by $A \sim B$ if there exists $C \in \mathrm{GL}_2(\mathbb{R})$ such that $A = CBC^{-1}$.

(a) Show that this is an equivalence relation
(b) Put a metric on $\mathrm{GL}_2(\mathbb{R})$ by saying that the distances between matrices A and B is $\max\{|a_{ij} - b_{ij}|\}$ where a_{ij} denotes the i, j entry of A. (you do not have to check that this defines a metric, but you can if you want to)
Show that the quotient $\mathrm{GL}_2(\mathbb{R})/\sim$ is not Hausdorff.
Possible hint: look at the equivalence class of $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

8. Define an equivalence relation on the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (remember, this has the subset topology from $\mathbb{R}^2$) by saying that $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = -x_2$ and $y_1 = -y_2$. (you do not have to check that this is an equivalence relation).
Show that the quotient $S^1/\sim$ is homeomorphic to the original space S^1.

9. **Challenge**, not to hand in:
Let's try to do the question above, but this time for the 2-dimensional sphere.
Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, and define two points to be equivalent if the vector representing one is -1 times the other. Is $S^2/\sim$ homeomorphic to S^2?
Think about how you would prove it, even if you don't have all the required tools yet.