Math 4530 Homework 8: homotopy

Point set topology is a disease from which the human race will soon recover.
- Henri Poincaré, 1908 (there is some debate on the phrasing of this, but he apparently really said it!)

Problems:

1. Let X, Y and Z be topological spaces. Suppose that $f : X \to Y$ is homotopic to $g : X \to Y$ and $k : Y \to Z$ is homotopic to $h : Y \to Z$. Show that $h \circ g$ is homotopic to $k \circ f$.

2. A space X is called *contractible* if the identity map $id : X \to X$ is homotopic to a constant map.
   (a) Prove that any contractible space is path-connected.
   (b) Prove that, if X is contractible, then any map $f : Y \to X$ is homotopic to a constant map

3. Recall that spaces X and Y are said to *have the same homotopy type* or to be *homotopy equivalent spaces* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity on Y and $g \circ f$ is homotopic to the identity on X.
   (a) Show that a contractible space has the same homotopy type as a point
   (b) Show that, if X and Y are homeomorphic, then they have the same homotopy type.
   (c) Show that "have the same homotopy type" is an equivalence relation on topological spaces.

4. (a) Following the strategy given in class on Thursday, show that the identity map $id : S^1 \to S^1$ is not homotopic to a constant map. (Or, in other words, the circle is not contractible.). **A suggested outline is given on the next page.**
   (b) Deduce that the letter P is not homotopic to the letter I.

5. BONUS: Is the torus $S^1 \times S^1$ homotopy equivalent to the sphere S^2? Why/why not?

Outline for question 4 (a)

Recall that we proved S^1 was homeomorphic to $\mathbb{R}/\sim$ where $x \sim y$ if $x = y$ or if $x = y + k$ for some $k \in \mathbb{Z}$. (In class we used $k \in 2\pi\mathbb{Z}$, but it doesn't matter which one you pick. On this homework we'll use $\mathbb{Z}$).

A *lift* of a continuous function $f : S^1 \to S^1$ is a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ \tilde{f}(x) = f \circ \pi(x)$ for all x.

(1) Let f be a continuous function $S^1 \to S^1$. Show that the set of lifts of f is in bijective correspondence with $\mathbb{Z}$.

(2) Let f be a continuous function $S^1 \to S^1$, and let $\tilde{f}$ be a lift of f.
Show that $\Delta(f) = \tilde{f}(0) - \tilde{f}(1)$ does not depend on which lift you chose.

(3) Let $F : [0,1] \times S^1 \to S^1$ be a continuous map. Show that there is a continuous map $\tilde{F} : [0,1] \times \mathbb{R} \to \mathbb{R}$ where, for each fixed $t \in [0,1]$ the function $\tilde{f}_t(x) = \tilde{F}(t,x)$ is a lift of $f_t(x) = F(t,x)$.

(4) Show that if $f : S^1 \to S^1$ is homotopic to $g : S^1 \to S^1$, then $\Delta(f) = \Delta(g)$, and conclude that the identity is not homotopic to a constant map.