HW 0: differential forms and manifolds self-assessment

MATH 6620 uses the language of differential forms and manifolds frequently. These exercises are a warm-up and a way to check if you're comfortable enough with this stuff. Ideally, it will take you < 1 hour to complete. But of course, life is not always ideal and you might be prompted to spend some time reviewing how differential forms work.

Bring a physical (on paper, not digital!) manifestation of your solutions to class on **Wednesday, Jan 25**.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}^4$ be given by $f(x_1, x_2) = (x_1x_2, \cos(x_1), 3x_2 + 17, 1)$. Compute the pullback of the following forms under f.
  (a) $(y_1 + y_2)dy_1 + e^{y_2}dy_2$
  (b) $dy_1 \wedge dy_3 + dy_2 \wedge dy_4$
  (c) $dy_1 \wedge dy_2 \wedge dy_3$

(2) (a) Let $\omega = x\,dy - y\,dx$, let $j : M \to \mathbb{R}^2$ be the inclusion of a bounded region with smooth boundary. Show that the area of M is equal to $\frac{1}{2}\int_{\partial M} j^*\omega$.
  (b) Generalize this to show how to compute the area of a region in $\mathbb{R}^n$ using the differential form

$$\sum_{i=1}^{n}(-1)^i x_i dx_1 \wedge dx_2 \wedge ... \wedge \widehat{dx_i} \wedge ... \wedge dx_n$$

(3) Let S^2 denote the unit sphere in $\mathbb{R}^3$. Give an example of a 2-form on S^2 that is nowhere zero. (there are many possibilities here, please explain your solution).

(4) Look at the 2-form you defined in the previous question. Is it closed? Is it exact? ("oh no, it looks really hard to tell if this is exact or not!" is an ok answer if you can explain why).

(5) Suppose M is a smooth manifold. What is a *vector field* on M? Answer by giving a plain-english description. (For instance, if the question was "what is the derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$?" you might say something like "the derivative is a collection of linear maps, one for each point in $\mathbb{R}^n$ and at each point it is the best local linear approximation to f"

(6) Suppose M is a smooth manifold. What is a *differential form* on M? Answer in plain english like the previous question.