Math 6620 Homework 1, due Monday January 30

Important info on HW: Homework is always due on Mondays, *on paper* at the beginning of class.

There are two types of problems. Type 1 are essential checks that you understand the material and notation and you should complete all of them. If you are totally following the lectures and the reading and understanding everything, they should not take much time. But, doing the reading and thinking about the lectures *does* take much time, so don't expect these to be totally effortless!

Type 2 questions are meant to make you curious. Each week, you should explain something you did on at least one of these problems. Maybe you have a proof, a good starting point, or you explore a special case. If you get really interested in a problem and do some outside reading, that's great too. However, anything you submit must be understandable by a student in our class. That means: use the notation from Lee, only use concepts that we have defined so far or that are covered in undergraduate calculus or the core graduate courses, etc. If you collaborate extensively, please indicate the other members of your group.

Reading this week: Lee, ch 2. You can skip the sections on pseudo-Riemannian and other generalizaions

Type 1 problems:

1. Consider the space $\{(x^1, x^2, \dots x^n, y) \in \mathbb{R}^{n+1} : y > 0\}$, with metric $ds^2 = \frac{(dx^1)^2+\dots+(dx^n)^2+dy^2}{y^2}$. We write (x, y) for $(x^1, x^2, \dots x^n, y)$ to be short.
   (a) For any given x, prove that the shortest length path between (x, a) and (x, b) is a vertical line. Your answer should involve integrals somewhere
   (b) Prove that the distance from any point to the boundary plane $y = 0$ is infinite.

2. Let N and S denote the north and south poles of S^2. Define a Riemannian metric on S^2 such that $d(N, S) > d(x, y)$ for all points $x, y \in S^2$ not equal to N, S. (It's ok to give a proof without intensive computation, as long as your argument is actually a proof). Now, modify your metric so that $d(N, S) = 27$.

3. Let $S^n(r)$ be the sphere of radius r in $\mathbb{R}^{n+1}$. After removing the one point from $S^n(r)$ where the last coordinate is r, we have a stereographic projection map ϕ to $\mathbb{R}^n$ defined by $\phi(y_1, \dots y_{n+1}) = \frac{1}{r-y_{n+1}}(y_1, \dots y_n)$. The *round metric* on the sphere is the induced metric from the Euclidean space $\mathbb{R}^{n+1}$. Compute the pullback of the round metric to $\mathbb{R}^n$ via ϕ.

Type 2 problems:

1. Suppose Γ_1 and Γ_2 are subgroups of translations of the Euclidean space $\mathbb{R}^n$, each isomorphic to $\mathbb{Z}^n$. When is $\mathbb{R}^n/\Gamma_1$ isometric to $\mathbb{R}^n/\Gamma_2$?

2. Given a Riemannian metric on M, we defined a distance $d(p, q)$ as an infininum of lengths of paths. Show that $d(p, q) = 0$ implies $p = q$. *(Note: if M is not Hausdorff this fails – can you give a counterexample? – so you should use Hausdorff somehow!)*

3. Let S_1 and S_2 be closed surfaces in $\mathbb{R}^3$. Let g_i be the metric on S_i induced from the Euclidean metric on $\mathbb{R}^3$. Suppose (S_1, g_1) and (S_2, g_2) are isometric. Is there necessarily an isometry of $\mathbb{R}^3$ taking S_1 to S_2? *Note:* this is not true for closed curves in $\mathbb{R}^3$, or even for closed curves in $\mathbb{R}^2$ (explain why?)
   Some ideas: is it easier if you assume S_1 is a round sphere? If you make other assumptions? Is "closed" really needed here? ...

4. Let (M, g) be a Riemannian n-manifold. Show that, near any point $p \in M$, there is a local orthonormal frame field (a set of n smooth vector fields, orthonormal at each point). *(Lee gives a not very detailed proof if you want hints)*
   Naively, you might try to solve this by making a coordinate chart such that the coordinate vector fields are orthonormal. As part of your solution, show that this naive approach is doomed: there is no such chart at any point of the round sphere. If a manifold does have a point p with such a chart, what can you say about a neighborhood of p?