<u>Recall</u> $M_1^n, M_2^m$ closed hyperbolic manifolds $n \geq 3$

$\varphi : \pi_1(M_1) \to \pi_1(M_2)$ isomorphism. There is a unique isometry $\phi : M_1 \to M_2$ s.t. $\phi_* = \varphi$.

<u>Step 1</u> $m = n$.

<u>Step 2</u> lift homotopy equivalence $f : M_1 \to M_2$ to $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ and extend $\tilde{f}$ to $F : S_\infty^{n-1} \to S_\infty^{n-1}$

<u>Step 3</u> Show $F$ is continuous & homeo.

<u>Step 4</u> Show $F$ is quasiconformal, ~~show $\tilde{f}$ are~~

<u>Step 5</u> Show $F$ is conformal. $(n=3)$.

<u>Step 6</u> Get isometry $\phi : M_1 \to M_2$ s.t. $\tilde{\phi}$ extends $F$, $\phi_* = \varphi$.

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$\partial \mathbb{H}^n$ is homeomorphic to $S_\infty^{n-1}$ which is smooth Riemannian manifold with usual round metric (gives a volume Borel measure as well)

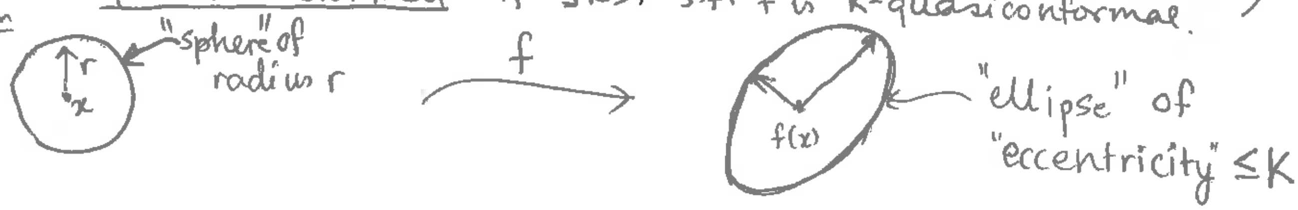
<u>Def</u> if $f$ is a homeomorphism of a metric space $X$ to $Y$. $f$ is called <u>K-quasiconformal</u> if for all $x \in X$

$$\limsup_{r \to 0} \frac{\sup\limits_{d(x,y)=r} d(f(x), f(y))}{\inf\limits_{d(x,y)=r} d(f(x), f(y))} \leq K$$

(note: Thurston's formula slightly different)

$f$ is <u>quasiconformal</u> if $\exists K \geq 1$ s.t. $f$ is K-quasiconformal.

<u>intuition</u>




(2) Observe: if $g$ is conformal then

$f$ is quasiconformal $\Leftrightarrow$ $g \circ f$ is so

$\Updownarrow$

$f \circ g$ is so.

Related notion

(1) Observe: if $f$ is a diffeo of Riemannian manifolds that preserves angles (i.e. $\dfrac{\langle Df(v), Df(w) \rangle}{\|Df(v)\| \|Df(w)\|} = \dfrac{\langle v, w \rangle}{\|v\| \|w\|}$)

(this is equivalent to $\underset{\substack{\text{matrix in}\\ \text{orthonormal frame}}}{[Df]} = \lambda A$ for some $\lambda \in \mathbb{R}^+$ & $A \in O(n)$ )

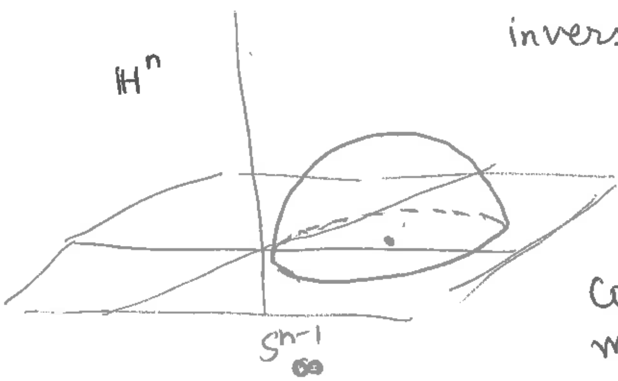
then $f$ is conformal. It is 1-quasiconformal.

④ Observation if $g : \mathbb{H}^n \to \mathbb{H}^n$ is an isometry then $G : S^{n-1}_\infty \to S^{n-1}_\infty$ is conformal.

Pf $\mathbb{H}^n$ in the ~~disk model~~ upper half-space model

Isom($\mathbb{H}^n$) is generated by inversions in spheres that are orthogonal to $S^{n-1}_\infty$.

This isometry extends to an inversion in a sphere in $S^{n-1}_\infty$.

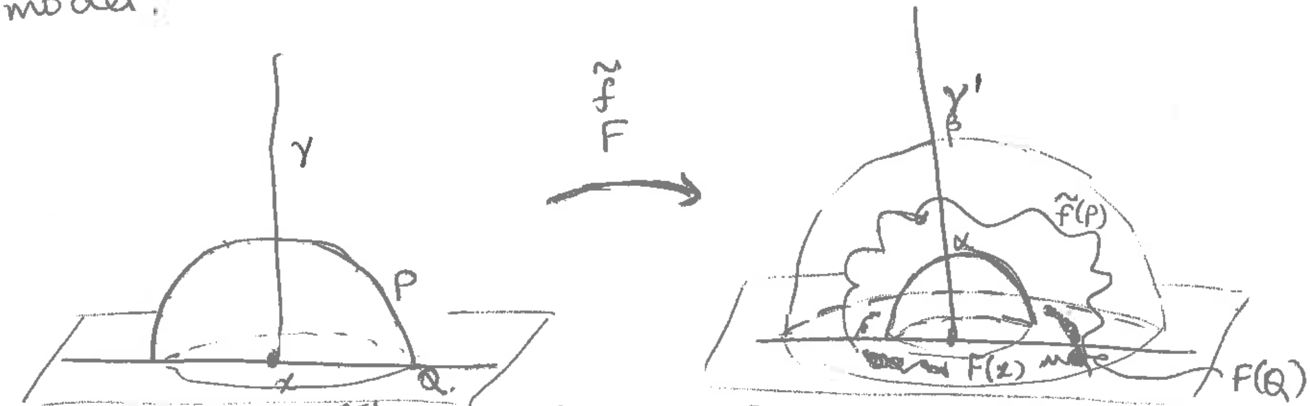



Check that inversions are (Riemannian) conformal, so are conformal.

Conversely every conformal map of $S^{n-1}_\infty$ extends to unique isometry of $\mathbb{H}^n$. ($n \geq 3$).

③ Observation (punctured) sphere $S^{n-1}$ is conformally equivalent to euclidean space $\mathbb{R}^{n-1}$ via stereographic projection (and this is the extension of the usual isometry $\mathbb{D}^n \to \mathbb{H}^n$).

To show $F$ is quasi-conformal use upper half space model.




A sphere $Q$ in $\mathbb{R}^{n-1}_\infty$ centred at $x$ is Euclidean sphere centred at $x$, which bounds a hyperplane $P$ in $\mathbb{H}^n$.

Let $\gamma$ be geodesic from $x$ to which is orthogonal to $P$ and $\gamma'$ be geodesic close to $\tilde{f}(\gamma)$. If required compose with an isometry of $\mathbb{H}^n$ / conformal map of $S^{n-1}_\infty$ so that $\gamma'$ is vertical geodesic at $F(x)$.

By previous lemma $\pi_{\gamma'}(\tilde{f}P)$ has diameter atmost $N$ $(=N(K,C))$ $\pi_{\gamma'}(P)$ is an interval on $\gamma'$ with (vertical) coordinates $[\alpha,\beta]$.

Then $N \geq \int_\alpha^\beta \frac{dt}{t} = \log \frac{\beta}{\alpha}$

$\tilde{f}(P)$ lies between the Euclidean spheres centred at $F(x)$ of radii $\alpha$ and $\beta$.

Thus $F(Q)$ lies between Euclidean spheres in $\mathbb{R}^{n-1}$ of radii $\alpha$ and $\beta$.

Therefore $\sup\limits_{y\in Q} d(F(x),F(y)) \leq \beta$, $\inf\limits_{y\in Q} d(F(x),F(y)) \geq \alpha$.

Hence ratio is $\frac{\beta}{\alpha} \leq e^N$

$\lim\limits_{\text{radius}(Q)\to 0} (\sup)$ ratio $\leq e^N$

Step 5

Theorem ① A K-quasiconformal map $F$ $S^{n-1}_\infty \to S^{n-1}_\infty$ is differentiable a.e. (for dim $\geq 2$) i.e. $n \geq 3$)

② $DF_x$ is nonsingular a.e.

③ $DF_x$ takes spheres to ellipsoids. if $0 \leq 1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1}$ are lengths of these axes then $\lambda_i$ are conformal invariants i.e. for any conf. diffeo $g$ $D(g \circ F)_x$ has lengths same as $DF_x$, and $D(F \circ g)_x$ same as

[ these are called singular values of $DF_x$ ]

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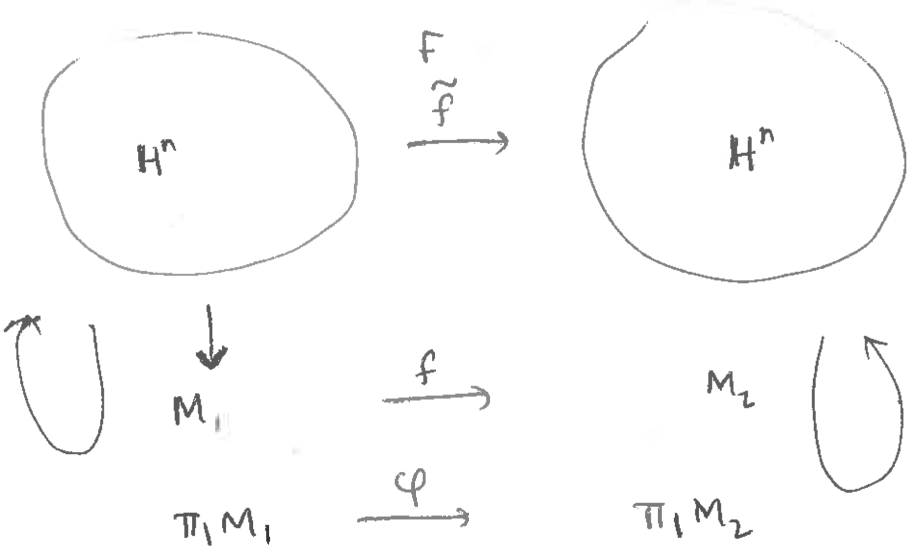
$\pi_1 M_1$ acts on $\mathbb{H}^n$ by deck transformations which are isometries.

Since isometries are in particular pseudo-isometries, they extend to homeomorphisms of $\partial \mathbb{H}^n = S^{n-1}_\infty$ which (saw before) are conformal.

Theorem $\pi_1 M_1$ acts ergodically on $S^{n-1}_\infty \times S^{n-1}_\infty$, and on $S^{n-1}_\infty$.

$\pi_1 M_2$ also acts on $\mathbb{H}^n$ and so ~~$\pi_1 M_1$~~ composing with $\varphi$, we get another action of $\pi_1 M_1$ on $\mathbb{H}^n$.




Since $f_* = \varphi$, $\tilde{f}$ is $\pi_1 M_1$-equivariant.

For any $g \in \pi_1 M_1$ and geodesic ray $\gamma$ in $\mathbb{H}^n$ with end $x$

$\tilde{f} \circ g = g \circ \tilde{f}$ so $g(\gamma)$ is geodesic ending at $g(x)$

and $\tilde{f} \circ g(\gamma) = g \circ \tilde{f}(\gamma)$ is quasi geodesic

is q.g. ending at $F(g(x))$

if $\tilde{f}(\gamma)$ is in R-nbd of $\gamma'$ then $g \circ \tilde{f}(\gamma)$ is in R-nbd of $g(\gamma')$ so has ~~geodesic~~ end $g(F(x))$

Thus $F$ is $\pi_1 M_1$ -equivariant.

Now $F \circ g = g \circ F$ so if $F$ is diff at $x$ then it is diff at $g(x)$ as well.

$$DF_{g(x)} \times Dg_x = Dg_{F(x)} \times DF_x.$$

As $g$ acts conformal on both domain & codomain $S^{n-1}_\infty$ $Dg_x$ & $Dg_{F(x)}$ are scalar multiples of orthogonal matrices.

Thus if $DF_x$ ~~has inv.~~ takes a sphere to ellipsoid $(\lambda_1, \dots, \lambda_{n-1})$ then $DF_{g(x)}$ takes $(Dg_x)^{-1}$ of that sphere to $(Dg_{F(x)})^{-1}$ of the ellipsoid $(\lambda_1, \dots, \lambda_{n-1})$

$\Rightarrow$ ellipsoid $(\mu\lambda_1, \dots, \mu\lambda_{n-1})$

Thus $e = \dfrac{\lambda_{n-1}}{\lambda_1}$ "maximum stretch" is a $\pi_1 M_1$ invariant function on $S^{n-1}_\infty$ (and is measurable)

$\Big[\ \lambda_{n-1} = \sup\limits_{\|v\|=1} \|DF_x(v)\|$ so is measurable, as partial derivatives $DF_x(v)$ are all measurable $\Big]$

Hence each level set of $e$ is $\pi_1 M_1$-invariant measurable subset of $S^{n-1}_\infty$ so has zero or full measure, ~~Only~~ and these union to all of $S^{n-1}_\infty$.

Thus one and only one has full measure

We show that $e=1$ a.e. If not then $e = k \cancel{\neq \ell}$, $k \neq 1$

<u>Case $n=3$</u> so $n-1=2$. $T_x S^2_\infty$ has two singular ~~$DF_x$ has~~ directions for $DF_x$, one of which is direction of maximal stretch. $\ell(x)$.

$\ell(x)$ is a line in $T_x S^2_\infty$ and is defined a.e. on $S^2_\infty$

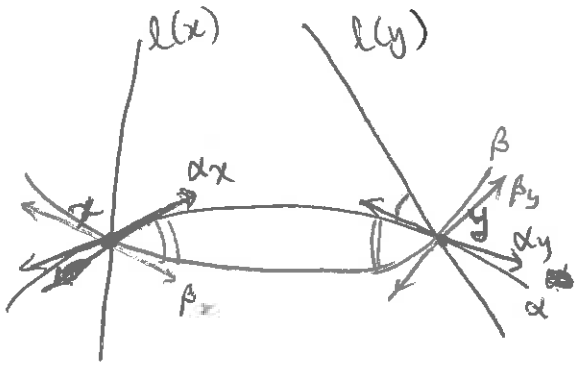
Since $F$ is $\pi_1 M_1$-equivariant $\ell(x)$ is $\pi_1 M_1$-equivariant ~~$\pi_1 M_1$-invariant~~.

(i.e. for $g \in \pi_1 M_1$, $(g_*)(\ell(x)) = \ell(gx)$)

($\pi_1 M_1$ acts on $TS^2_\infty$ by pushforward)

Define angle between $\ell(x)$, $\ell(y)$ as follows

for any arc $\alpha$ of circle going through $x$ and $y$ let tangent lines to $\alpha$ at $x, y$ be $\alpha_x$, $\alpha_y$




$$\theta(x,y) = \angle(\ell(y), \alpha_y) + \angle(\ell(x), \alpha_x)$$

$\downarrow$

angle "counterclockwise" that $\ell(y)$ must be rotated by to coincide with $\alpha_y$.

$\theta : S^2_\infty \times S^2_\infty \to \mathbb{R}/\pi\mathbb{R}$ is defined a.e.

independent of $\alpha$

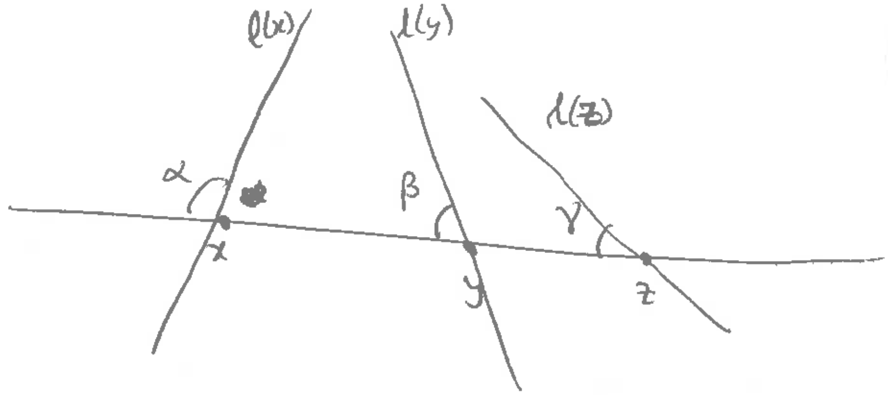
Since $\pi_1 M_1$ acts on $S^2_\infty$ conformally taking circles to circles, $\theta$ is $\pi_1 M_1$-invariant.

As $\pi_1 M_1 \circlearrowright S^2_\infty \times S^2_\infty$ is ergodic, $\theta$ is constant a.e.

But this is not possible

~~Work is in the conformal model of the plane.~~

let $x, y, z$ be three points on a line at ~~for~~ which $F$ is diff.




then $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$ are all $\theta_0 \pmod{\pi}$

Thus $2\alpha$ is $\theta_0 \pmod{\pi}$

$\alpha$ is $\theta_0/2 \pmod{\pi/2}$.

That is $l(x)$ makes an angle of $\theta_0/2$ with almost every geodesic through $x$, a contradiction.

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Thus $e=1$ a.e.

<u>Thm</u> if $F$ is diff. almost everywhere and $DF_x$ is a conformal matrix a.e. then $F$ is diff. and conformal.

<u>Thm</u> if $F: S^{n-1}_\infty \to S^{n-1}_\infty$ is conformal then there is unique isometry $f: \mathbb{H}^n \to \mathbb{H}^n$ $(n \geq 3)$ such that $f$ extends to $F$.

<u>Proof:</u> of uniqueness. For $p \in \mathbb{H}^n$ pick two distinct geodesics intersecting at $p$. The images of geodesics are geodesics under isometry, hence are determined by their endpoints det. by $F$. Thus image of $p$ is determined as well.

Thus there is isometry $\Phi : \mathbb{H}^n \to \mathbb{H}^n$ such that $\Phi$ extends to $F$.

For any $g$ in $\pi_1 M_1$ $\Phi \circ g$ extends to $F \circ g = g \circ F$
and $g \circ \Phi$ extends to $g \circ F$

Thus we must have (by uniqueness)

$$\Phi \circ g = g \circ \Phi .$$

Thus $\Phi$ descends to an ~~diff~~ isometry $M_1 \xrightarrow{\phi} M_2$
and satisfies $\phi_* = \varphi$.