NOTES ON THE REGULARITY OF QUASICONFORMAL HOMEOMORPHISMS

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1. Introduction

The purpose of these notes is to give a self-contained proof of the following theorem,

Theorem 1.1. *Let $f : S^n \to S^n$ be a K-quasiconformal homeomorphism, $n \geq 2$, $K \geq 1$. Then*

(1) *f is differentiable a.e.*
(2) *The differential Df_x is nonsingular at a.e. x at which f is differentiable.*
(3) *Let x be a point at which Df_x is nonsingular, and let $v, w \in T_xS^n$ with $\|v\| = \|w\| \neq 0$. Then*
$$\frac{1}{K} \leq \frac{\|Df_x(v)\|}{\|Df_x(w)\|} \leq K$$

An analogous theorem holds for quasi-conformal homeomorphisms between subdomains of S^n, as can easily be seen from the fact that all of the above statements are local in nature. Here a.e. refers to the Lebesgue measure on S^n. Throughout these notes, all measures will be Borel measures on the appropriate topological space, which will always be some Borel subset of $\mathbb{R}^n$ for some n. "Almost everywhere" will always refer to the Lebesgue measure on this subset of the appropriate dimension, unless otherwise noted. Lebesgue measure on $\mathbb{R}^n$ will be denoted by m_n. When there is no possibility of ambiguity, we will simply write m.

2. Background material in Analysis

We review some basic theorems from real analysis that will be used frequently. The proofs are all standard and hence mostly omitted.

2.1. **Absolute continuity.** A function $F : [a, b] \to \mathbb{R}^m$ $(-\infty \leq a \leq b \leq \infty)$ is called *absolutely continuous* if for every $\varepsilon > 0$, there is some $\delta > 0$ such that for any positive integer N, and any collection of N disjoint subintervals of $[a, b]$, written (a_j, b_j), $1 \leq j \leq N$, we have

$$\sum_{j=1}^{N} |b_j - a_j| < \delta \;\Rightarrow\; \sum_{j=1}^{N} |F(b_j) - F(a_j)| < \varepsilon \tag{2.1}$$

Given a measurable space $(X, \mathcal{A})$, and two σ-finite measures μ, ν on the σ-algebra $\mathcal{A}$ of measurable sets, we say that ν is *absolutely continuous* with respect to μ, written $\nu \ll \mu$, if $\mu(E) = 0$ implies that $\nu(E) = 0$. In this case, there is a function $g \in L^1(X, \mathcal{A}, \mu)$, uniquely defined μ-a.e., such that

$$\nu(E) = \int_E g \, d\mu \tag{2.2}$$

g is called the Radon-Nikodym derivative of ν with respect to μ, and is sometimes written $\frac{d\nu}{d\mu}$.

On $\mathbb{R}$, these two notions of absolute continuity are closely connected by a theorem which is sometimes called "the fundamental theorem of calculus for Lebesgue integrals". For a closed interval $[a, b]$, $-\infty < a < b < \infty$, and a continuous function $F : [a, b] \to \mathbb{R}$, we can define a signed Borel measure μ on $[a, b]$ by

$$\mu([a, x]) = F(x) - F(a) \tag{2.3}$$

for $a \leq x \leq b$.

Theorem 2.1. *For a continuous function $F : [a, b] \to \mathbb{R}$, the following are equivalent,*

1. *F is absolutely continuous.*
2. *There is a function $f \in L^1(m)$ such that*
$$F(x) - F(a) = \int_a^x f \, dm$$
3. *F is differentiable a.e., $F' \in L^1(m)$, and*
$$F(x) - F(a) = \int_a^x F' \, dm$$
4. *The signed Borel measure μ is absolutely continuous with respect to m.*

Hence absolutely continuous functions are precisely the class of functions for which the fundamental theorem of calculus holds true, in the sense that we can integrate the derivative and recover the function.

Clearly any absolutely continuous function is uniformly continuous, by taking $N = 1$ in the definition. However, there are uniformly continuous functions which are not absolutely continuous. One example is the Cantor function on $[0, 1]$, also known as the devil's staircase.

2.2. **Lebesgue Density Theorem.** For any $n \in \mathbb{N}$, let $f \in L^1_{loc}(\mathbb{R}^n, m)$ be a locally Lebesgue-integrable function. Let $B(r, x)$ denote the ball of radius r centered at x. A point $x \in \mathbb{R}^n$ is said to be a *Lebesgue point* for f if

$$\lim_{r \to 0} \frac{1}{m(B(r, x))} \int_{B(r,x)} |f(y) - f(x)| \, dy = 0 \tag{2.4}$$

The Lebesgue density theorem implies that Lebesgue density points for f are abundant.

Theorem 2.2. *Almost every $x \in \mathbb{R}^n$ is a Lebesgue point for f.*

A particularly useful consequence is that

$$\lim_{r \to 0} \frac{1}{m(B(r, x))} \int_{B(r,x)} f(y) \, dy = f(x) \tag{2.5}$$

for a.e. $x \in \mathbb{R}^n$, since this is true for every Lebesgue point.These theorems are true if we replace the balls $B(r,x)$ with any other family of sets parametrized by r which shrinks to x sufficiently nicely, e.g. cubes centered at x of side length r.

Let E be a Borel subset of $\mathbb{R}^n$. We say that $x \in E$ is a Lebesgue density point for E if

$$\lim_{r \to 0} \frac{m(B(r,x) \cap E)}{m(B(r,x))} = 1 \tag{2.6}$$

By applying the Lebesgue density theorem to the characteristic function for E, we see that a.e. $x \in E$ is a Lebesgue density point for E.

Let $F : U \to V$ be a homeomorphism between two open subsets of $\mathbb{R}^n$. We can define a new measure ν_F on U by $\nu_F(A) = m(F(A))$ for a measurable $A \subset U$. We define the *volume derivative* of F at x as

$$\nu_F'(x) := \lim_{r \to 0} \frac{\nu_F(B(r,x))}{m(B(r,x))} \tag{2.7}$$

The Lebesgue density theorem implies that this limit exists for a.e. $x \in U$, and that $\nu_F' \in L^1_{loc}(m)$. In fact, ν_F' is the Radon-Nikodym derivative of the absolutely continuous part of the measure ν_F (with respect to m). Suppose that $m(A) = 0$ implies that $m(F(A)) = 0$. Then ν_F is absolutely continuous with respect to m, ν_F' is the Radon-Nikodym derivative of ν_F, and

$$m(F(A)) = \int_A \nu_F' \, dm \tag{2.8}$$

It is easily checked that if F is differentiable at a point x, then $\nu_F'(x)$ is the absolute value of the Jacobian of F at x. Hence the volume derivative is a generalization of the Jacobian of a differentiable mapping.

If we forget the homeomorphism F, we can still define, for any regular Borel measure μ, the volume derivative

$$\mu'(x) = \lim_{r \to 0} \frac{\mu(B(r,x))}{m(B(r,x))} \tag{2.9}$$

(a *regular* Borel measure is a Borel measure which takes finite values on compact sets) For regular Borel measures, the volume derivative μ' exists a.e. and is an $L^1_{loc}(m)$ function. In particular, it is finite a.e.

2.3. **Egoroff's Theorem.** A classic, extremely useful theorem of analysis is Egoroff's theorem. Let $(X, \mathcal{A}, \mu)$ be a measure space.

Theorem 2.3. *Suppose $\mu(X) < \infty$, and $(f_n)_{n \geq 1}$ is a sequence of measurable real-valued functions converging pointwise a.e. to a function f. Then for every $\varepsilon > 0$, there exists $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \backslash E$.*

2.4. **Hausdorff measure.** We will need a natural way of assigning a 1-dimensional measure to segments of paths in $\mathbb{R}^n$, for any $n \geq 1$. The most natural way to do this is via the 1-dimensional

Hausdorff measure, which we will denote by Λ. Fix n. We define, for any $A \subset \mathbb{R}^n$,

$$\Lambda(r, A) = \inf \left\{ \sum_1^\infty \mathrm{diam} B_j : \ A \subset \bigcup_1^\infty \text{ and } \mathrm{diam} B_j \leq r \right\} \tag{2.10}$$

$$\Lambda(A) = \lim_{r \to 0} \Lambda(r, A) \tag{2.11}$$

The sets B_j in the definition of $\Lambda(r, A)$ may be restricted to the class of open subsets of $\mathbb{R}^n$. Λ is a metric outer measure on $\mathbb{R}^n$, hence all Borel sets are Λ-measurable and therefore the restriction of Λ to the Borel σ-algebra is a Borel measure. Λ is invariant under isometries of $\mathbb{R}^n$. For $n = 1$, Λ agrees with the usual Lebesgue measure on $\mathbb{R}$.

It is an easy exercise to see that if P is orthogonal projection onto a 1-dimensional subspace of $\mathbb{R}^n$, and $A \subset \mathbb{R}^n$ is Borel measurable, then $m_1(P(A)) = \Lambda(P(A)) \leq \Lambda(A)$. Now let $A \subset \mathbb{R}^n$ be a *connected* Borel measurable subset, and let $x, y \in A$. Let ℓ be a straight line segment joining x to y in $\mathbb{R}^n$, and let L be the line in $\mathbb{R}^n$ containing ℓ. The orthogonal projection of A onto L contains the line segement ℓ, since A is connected, and we thus obtain the fundamental diameter estimate

$$\mathrm{diam}(A) \leq \Lambda(A) \tag{2.12}$$

We will use estimates on the Hausdorff measure of the images of Lebesgue-small sets to prove the absolute continuity of maps $F : [a, b] \to \mathbb{R}^n$. More precisely,

Lemma 2.4. *Suppose $F : [a, b] \to \mathbb{R}^n$ is an injective map such that for every $\varepsilon > 0$, there is $\delta > 0$ such that whenever $I_1, \ldots, I_k$ are disjoint closed subintervals of $[a, b]$ satisfying $\sum_{j=1}^n m_1(I_j) < \delta$, we have $\sum_{j=1}^n \Lambda(F(I_j)) < \varepsilon$. Then F is absolutely continuous.*

The proof is immediate from the diameter estimate (2.12).

3. The differentiability of quasiconformal mappings

The goal of this section is to prove the first part of Theorem 1.1, which states that a quasiconformal homeomorphism of S^n is differentiable a.e. for $n \geq 2$. We first define what it means for a homeomorphism to be quasiconformal. Since all of the statements of theorem 1.1 are local, from now on all homeomorphisms f will be between bounded open subsets of $\mathbb{R}^n$, and all distances will be measured in the Euclidean metric, which is locally comparable to any Riemannian metric we would put on S^n. So fix from now on a homeomorphism $f : U \to V$, where U, V are bounded open subsets of $\mathbb{R}^n$. Define

$$D_I(x, r) = \inf_{|x-y|=r} |f(x) - f(y)| \tag{3.1}$$

$$D_O(x, r) = \sup_{|x-y|=r} |f(x) - f(y)| \tag{3.2}$$

$$D(x) = \limsup_{r \to 0} \frac{D_O(x, r)}{D_I(x, r)} \tag{3.3}$$

The first two functions D_I and D_O measure the shortest and longest distance from the center of the image of a sphere centered at x. The ratio of these two functions for a particular r is a measure of the extent to which this image deviates from being a sphere centered at $f(x)$. $D(x)$ is the infinitesmal analogue of this measure. The homeomorphism f is called K-quasiconformal if there is a constant $K \geq 1$ such that $D(x) \leq K$ for every $x \in U$. One may define a conformal mapping as a map which infinitesmally maps spheres to spheres in the above sense, and we then see that $D(x)$ measures the failure of f to be conformal at x. We also see that a quasiconformal homeomorphism is a homeomorphism which fails to be conformal only by a uniformly bounded amount.

It is also convenient to introduce the function

$$H(x) = \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} \tag{3.4}$$

3.1. **The Rademacher-Stepanov Theorem.** The analytic backbone of this differentiability result is the following theorem of Rademacher-Stepanov, which states that under certain conditions, a continuous function which has partial derivatives a.e. is differentiable a.e.

Theorem 3.1. *Suppose that U is an open set in $\mathbb{R}^n$ and that $f : U \to \mathbb{R}^m$ is a map such that*

1. *f is continuous.*
2. *The partial derivatives of f exist a.e.*
3. *$H(x) < \infty$ a.e.*

Then f is differentiable a.e.

The conclusion of this theorem is true even if hypotheses (1) and (2) are dropped, but we will not need the result in this generality. Partial derivatives refers here only to the derivatives of f along directions parallel to the n coordinate axes. Before proceeding with the proof of the Rademacher-Stepanov theorem, we outline how it will be used to prove the differentiability of the K-quasiconformal homeomorphism f. The Rademacher-Stepanov theorem implies that it suffices to show that f has partial derivatives a.e. and that $H(x) < \infty$ a.e. The first of these is a consequence of a remarkable regularity property of quasiconformal homeomorphisms; quasiconformal maps are absolutely continuous along a.e. line segment parallel to the coordinate axes in $\mathbb{R}^n$. The second condition is an easy consequence of the Lebesgue density theorem.

Proof. By considering the coordinate functions of f separately and observing that the assumptions on f are local in nature, it suffices to assume that $m = 1$ and U is a bounded open subset of $\mathbb{R}^n$. The idea of the proof is relatively simple; we simply follow the proof that a continuous function with continuous partial derivatives is differentiable. The difficulty lies in finding a proper full measure set of U on which this argument will work.

Fix $\delta > 0$. For $c \in \mathbb{N}$, let A_c denote the set of all points $x \in U$ such that $|f(x+h) - f(x)| \leq c|h|$ whenever $|h| < 1/c$ and $x \in U$. Then $A_c \subset A_{c+1}$, and

$$\bigcup_{c=1}^{\infty} A_c = \{x \in U : H(x) < \infty\} \tag{3.5}$$

is a full measure subset of U by assumption. Hence we find a large enough integer k such that $m(U\backslash A_k) < \delta/2$. We also define functions

$$g_j(x) = \sup_{1\le i\le n} \sup_{0<|r|<1/j} \left|\frac{f(x+re_i)-f(x)}{r} - \partial_i f(x)\right| \tag{3.6}$$

Since the partial derivatives of f exist a.e. in U, g_j is defined a.e. in U for each j, and $g_j \to 0$ a.e. It is clear also that g_j is measurable. Hence by Egoroff's theorem, there is a compact set $F \subset U$ such that $m(U\backslash F) < \delta/2$ and such that $g_j|_F \to 0$ uniformly. Let E be the set of Lebesgue density points of $F \cap A_k$. By the Lebesgue density theorem, E has full measure in $F \cap A_k$, and so $m(U\backslash E) < \delta$. Note that every point of E is itself a density point of E. We will prove that f is differentiable at every point of E. Since $\delta > 0$ was arbitrary, this proves that f is differentiable at a.e. point in U.

Let $y \in E$, and let $t > 0$ be small enough that $B(y, 2t) \subset U$. Let x be some point in U such that $|x - y| = t$. Define

$$z^i = (x_1, \dots, x_i, y_{i+1}, \dots, y_n)$$

for $0 \le i \le n$, so that $z^0 = y$ and $z^n = x$. The points z^i need not lie in E. However, using the fact that y is a density point for E, we will be able to find points u^i sufficiently close to these z^i which do lie in E. More precisely, for each $t > 0$ sufficiently small, we will find $r = r(t) \le t$ such that $B(z^i, r) \cap E$ is nonempty. Further, $r(t)$ can be chosen so that $r(t)/t \to 0$ as $t \to 0$.

The idea is to find r such that the pair of inequalities

$$m(B(z^i, r)\backslash E) \le m(B(y, 2t)\backslash E) < m(B(z^i, r)) \tag{3.7}$$

hold. Clearly the first inequality is satisfied if $r \le t$. The second inequality is satisfied if

$$r > 2t \left(\frac{m(B(y, 2t)\backslash E)}{m(B(y, 2t))}\right)^{1/n} \tag{3.8}$$

as we may write

$$m(B(y, 2t)\backslash E) = m(B(y, 2t)) \left(\frac{m(B(y, 2t)\backslash E)}{m(B(y, 2t))}\right) < \frac{r^n}{(2t)^n} m(B(y, 2t)) = m(B(y, r)) \tag{3.9}$$

and all balls of radius r have the same measure. We claim that for sufficiently small t, we may choose any r satisfying

$$2(t + t^2) \left(\frac{m(B(y, 2t)\backslash E)}{m(B(y, 2t))}\right)^{1/n} > \ r \ > 2t \left(\frac{m(B(y, 2t)\backslash E)}{m(B(y, 2t))}\right)^{1/n} \tag{3.10}$$

It is easy to check that the left side of (3.10) is $\le t$ for t small enough, using that y is a density point of E. To be definite, we can choose $r(t)$ to be the average of each side of (3.10). Then it is immediate as well that $r(t)/t \to 0$ as $t \to 0$.

Knowing now that we can choose $r(t)$ dependent on t as we claimed, we will suppress the argument of $r(t)$ and just write r. We can find $u^i \in B(z^i, r) \cap E$ for each i, since this set is nonempty by our choice of r. Put $u^0 = y$. Let $v^i = u^{i-1} + (x_i - y_i)e_i$, and write

$$f(x) - f(y) = f(x) - f(u^n) + \sum_{i=1}^{n} (f(u^i) - f(v^i)) + \sum_{i=1}^{n} (f(v^i) - f(u^{i-1})) \tag{3.11}$$

Recall that the compact set F was chosen so that $g_j \to 0$ uniformly on F. This implies that the functions $\partial_i f$ are well-defined and are uniformly continuous on F for each i. Since $E \subset F$, the same holds for E. From equation (3.11), we have the estimate

$$\left| f(x) - f(y) - \sum_{i=1}^{n} \partial_i f(y)(x_i - y_i) \right| \leq |f(x) - f(u^n)| + \sum_{i=1}^{n} |f(u^i) - f(v^i)| \tag{3.12}$$

$$+ \sum_{i=1}^{n} \left| f(v^i) - f(u^{i-1}) - \sum_{i=1}^{n} \partial_i f(u^{i-1})(x_i - y_i) \right| \tag{3.13}$$

$$+ \sum_{i=1}^{n} |\partial_i f(u^{i-1}) - \partial_i f(y)||x_i - y_i| \tag{3.14}$$

To conclude the proof, we show that each of the three lines above is bounded by a function of the form $\varepsilon(t)$, where $\varepsilon(t)/t \to 0$ as $t \to 0$ which implies that f is differentiable at y.

For line (3.12), recall that $E \subset A_k$, and so if we take $t < 1/(2k)$, then $|f(p+h) - f(p)| \leq k|h|$ for any $p \in E$. By the construction of the points u^i, this implies that

$$|f(x) - f(u^n)| + \sum_{i=1}^{n} |f(u^i) - f(v^i)| \leq k(1+2n) \cdot r(t) \tag{3.15}$$

since

$$|v^i - u^i| \leq |u^{i-1} - z^{i-1}| + |u^i - z^i| \leq 2r \tag{3.16}$$

as may be checked from how the points v^i are chosen. Since $r(t)/t \to 0$ as $t \to 0$, this takes care of the first line.

For line (3.13), observe that

$$\sum_{i=1}^{n} \left| f(v^i) - f(u^{i-1}) - \sum_{i=1}^{n} \partial_i f(u^{i-1})(x_i - y_i) \right| \leq t \cdot \sum_{i=1}^{n} g_{1/t}(u^{i-1}) \leq nt \cdot \sup_{u \in F} g_{1/t}(u) \tag{3.17}$$

and $\sup_{u \in F} g_{1/t}(u) \to 0$ as $t \to 0$ since $g_j \to 0$ uniformly on F as $j \to \infty$.

Lastly, for line (3.14), the functions $\partial_i f$ are uniformly continuous on E as remarked earlier, and so

$$\frac{1}{t} \sum_{i=1}^{n} |\partial_i f(u^{i-1}) - \partial_i f(y)||x_i - y_i| \leq n \left(\sup_{1 \leq i \leq n} \sup_{\substack{|u-v|<t \\ u,v \in E}} |\partial_i f(u) - \partial_i f(v)| \right) \to 0 \tag{3.18}$$

as $t \to 0$. □

3.2. **The ACL Property.** We say that a homeomorphism $f : U \to V$ of open sets is ACL (absolutely continuous on lines) if for every compact n-box $Q = \{x : a_i \leq x_i \leq b_i\}$, the restrictions of f to the line segments parallel to the coordinate axes which lie inside of $Q \cap U$ are absolutely continuous, for almost every line segment. To make this assertion more precise, let P_i be the orthogonal projection of $\mathbb{R}^n$ onto the ith coordinate plane $\{x_i = 0\}$. Then f is ACL if for each i, $1 \leq i \leq n$, and each compact n-box Q, the set of $y \in P_i(U \cap Q)$ such that the map $t \to f(y + te_i)$ is *not* absolutely continuous ($a_i \leq t \leq b_i$) has Lebesgue m_{n-1} measure zero in $\{x_i = 0\}$.

Proposition 3.2. *Suppose that f is ACL. Then f has partial derivatives a.e. in U.*

Proof. For each i, $1 \leq i \leq n$, let A_i be the set of points $z \in U$ such that the ith partial derivative of f does not exist at z. It is easy to check that A_i is a Borel subset of U. Let $Q = \{x : a_i \leq x_i \leq b_i\}$ be a compact n-box. By Fubini's theorem,

$$m_n(Q \cap A_i) = \int_{P_i(Q)} m_1(P_i^{-1}(y) \cap A_i \cap Q)\, dm_{n-1}(y) \tag{3.19}$$

where m_1 here is 1-dimensional Lebesgue measure on the fiber $P_i^{-1}(y)$, and $y \in P_i(Q)$. For a.e. $y \in P_i(Q)$, f is absolutely continuous $P_i^{-1}(y)$, hence by Theorem 2.1, $f|P_i^{-1}(y)$ is differentiable a.e. on $P_i^{-1}(y)$, and therefore $m_1(P_i^{-1}(y) \cap A_i \cap Q) = 0$. This implies that $m_n(Q \cap A_i) = 0$. Since this holds for every n-box Q, we conclude that $m_n(A_i) = 0$. □

Hence it suffices to prove that any quasiconformal homeomorphism has the ACL property, in order to prove that partial derivatives exist a.e. We will do this by estimating the 1-dimensional Hausdorff measure Λ of the images under f of line segments parallel to the coordinate axes. We will then use Lemma 2.4 to derive absolute continuity on a full measure subset of line segments.

We first need a covering lemma for compact subsets of $\mathbb{R}$.

Lemma 3.3. *Suppose that F is a compact set in $\mathbb{R}$ and that $\varepsilon > 0$. Then there exists $\delta > 0$ with the following property: For every $r \in (0, \delta)$, there exists a finite covering of F with open intervals $I_1, \dots, I_p$ such that*

(1) *$m(I_j) = 2r$ for $1 \leq j \leq p$.*
(2) *The center of I_j belongs to F*
(3) *Each point of F belongs to at most two intervals I_j*
(4) *$pr < m(F) + \varepsilon$*

Proof. Choose an open set G such that $F \subset G$ and $m(G) < m(F) + \varepsilon$. Set $\delta = d(F, \mathbb{R} \backslash G)$. Suppose that $0 < r < \delta$. For $x \in F$ put $I_j(x) = (x - r, x + r)$. This gives a covering of F by open sets, and extracting a finite cover gives a covering of F which satisfies (1) and (2). By removing redundant intervals, it is easy to check that (3) can be satisfied as well.

Now let $I_1, \dots I_p$ be our covering satisfying (1),(2), and (3). We claim that it then satisfies (4) as well. Since every point of F lies in at most two intervals from this covering, and since each $I_j \subset G$,

$$pr = \frac{1}{2} \sum m(I_j) \leq m(G) < m(F) + \varepsilon \tag{3.20}$$

□

Let Ω_n denote the volume of an n-ball of radius 1, so that an n-ball of radius r has volume $\Omega_n r^n$.

Theorem 3.4. *Let $f : U \to V$ be a quasiconformal homeomorphism, . Then f is ACL.*

Proof. Let $Q = \{x : a_i \leq x_i \leq b_i\}$ be compact n-box contained inside U. Fix some i, $1 \leq i \leq n$, from now on, as the argument will be the same for each coordinate. Identify $\{x_i = 0\}$ with $\mathbb{R}^{n-1}$, and let P denote the projection onto $\{x_i = 0\}$

Define a Borel measure μ on $\mathbb{R}^{n-1}$ by (for $A \subset \mathbb{R}^{n-1}$)

$$\mu(A) = m_n(f(Q \cap P^{-1}(A))) \tag{3.21}$$

It's easy to see that μ is a regular Borel measure on $\mathbb{R}^{n-1}$, hence its volume derivative μ' is finite a.e. Fix some $y \in \mathbb{R}^{n-1}$ such that $\mu'(y) < \infty$, and set $J = Q \cap P^{-1}(y)$. We claim that f is absolutely continuous on this line segment. Since $\mu' < \infty$ a.e. in $\mathbb{R}^{n-1}$, this will prove the theorem.

Let F be a compact subset of $J \cap \operatorname{int}(Q)$. Our goal is to estimate $\Lambda(f(F))$ in terms of $m_1(F)$, in order to prove that if $m_1(F)$ is small, then $\Lambda(f(F))$ is small. Choose some K such that $D(x) < K$ for every $x \in U$. Consider k large enough that Q contains a $1/k$ neighborhood of F, and let F_k be the set of all $x \in F$ such that $0 < r < 1/k$ implies $D_O(x, r) \leq K D_I(x, r)$. Then $F_k \subset F_{k+1}$, $F = \cup F_k$, and each F_k is compact by the continuity of f.

We now fix k, and focus on estimating $\Lambda(f(F_k))$. Choose $\varepsilon > 0$, $t > 0$, and let $\delta > 0$ be given by Lemma 3.3 for the set F_k. Choose $r > 0$ small enough that $r < \min(\delta, 1/k)$ and such that $|f(x) - f(z)| < t$ whenever $x, z \in Q$ and $|x - z| \leq 2r$. Let $I_1, \ldots, I_p$ be the covering of F_k from Lemma 3.3. Let $x_j \in F_k$ be the center of I_j. We are at last ready to begin making estimates.

We first get a preliminary estimate on $\Lambda(t, f(F_k))$ in terms of the function $D_O(-, r)$. Let $A_j = B(x_j, r)$ be the open n-ball centered at x_j with radius r, whose intersection with J is the interval I_j. Since $f(F_k) \subset \cup f(A_j)$, and $\operatorname{diam}(f(A_j))$, we have

$$\Lambda(t, f(F_k)) \leq \sum_{j=1}^{p} \operatorname{diam}(f(A_j)) \leq 2 \sum_{j=1}^{p} D_O(x_j, r) \tag{3.22}$$

At this point in the proof, quasiconformality still has not entered the picture. Continuing, we can think of the rightmost sum in 3.22 (ignoring the factor of 2) as $\sum_{j=1}^{p} 1 \cdot D_O(x_j, r)$. Applying Hölder's inequality with conjugate exponents n and $n/(n-1)$, and then applying part (4) of Lemma 3.3, we have the bound

$$\left(\sum_{j=1}^{p} 1 \cdot D_O(x_j, r) \right)^n \leq \left(\sum_{j=1}^{p} 1^{n/(n-1)} \right)^{n-1} \left(\sum_{j=1}^{p} D_O(x_j, r)^n \right) \tag{3.23}$$

$$= p^{n-1} \sum_{j=1}^{p} D_O(x_j, r)^n \tag{3.24}$$

$$\leq (m_1(F_k) + \varepsilon)^{n-1} \frac{1}{r^{n-1}} \sum_{j=1}^{p} D_O(x_j, r)^n \tag{3.25}$$

$$\leq (m_1(F) + \varepsilon)^{n-1} \frac{1}{r^{n-1}} \sum_{j=1}^{p} D_O(x_j, r)^n \tag{3.26}$$

(since $F_k \subset F$). We now finally need to use quasiconformality, to control the term $\frac{1}{r^{n-1}} \sum_{j=1}^{p} D_O(x_j, r)^n$ as $r \to 0$. Our heuristic is the following: if f does not distort the shape of small n-balls too much, then the sum $\sum_{j=1}^{p} D_O(x_j, r)^n$ should be proportional to the volume of $f(Q \cap P^{-1}(W))$, where $W = B^{n-1}(y, r)$ is an $(n-1)$-ball around y in $\mathbb{R}^{n-1}$ of radius r. In turn, $f(Q \cap P^{-1}(W))$ should be proportional by some distortion factor to the length of the path $f(J)$ (a finite constant!) multiplied by $m_{n-1}(W) = \Omega_{n-1} r^{n-1}$, if we think of $f(Q \cap P^{-1}(W))$ as a cylinder surrounding $f(J)$. Hence $\sum_{j=1}^{p} D_O(x_j, r)^n$ is on the order of r^{n-1} (independent of p!).

All that remains is to make the above argument rigorous. Since each $x_j \in F_k$, we have $D_O(x_j, r) \leq KD_I(x_j, r)$ for each j. Observe that $f(\partial A_j)$ is the homeomorphic image of an $n-1$ sphere in $\mathbb{R}^n$, hence by the higher-dimensional Jordan curve theorem, it cuts $\mathbb{R}^n$ into components, one bounded, one unbounded. The bounded component contains $f(x_j)$, and in fact contains the open ball $B(f(x_j), D_I(x_j, r))$, by the definition of D_I. Since f is a homeomorphism, f must map the open ball A_j homeomorphically onto the bounded component of $\mathbb{R}^n \backslash f(\partial A_j)$. In particular, we conclude that $B(f(x_j), D_I(x_j, r)) \subset f(A_j)$. This implies that

$$\sum_{j=1}^{p} D_O(x_j, r)^n \leq K^n \sum_{j=1}^{p} D_I(x_j, r)^n \leq \frac{K^n}{\Omega_n} \sum_{j=1}^{p} m(f(A_j)) \tag{3.27}$$

By construction of the intervals I_j, each point of $Q \cap P^{-1}(W)$ lies in at most two n-balls A_j. Hence each point of $f(Q \cap P^{-1}(W))$ lies in at most two of the sets $f(A_j)$. This gives us the estimate

$$\frac{K^n}{\Omega_n} \sum_{j=1}^{p} m(f(A_j)) \leq 2\frac{K^n}{\Omega_n} m(f(Q \cap P^{-1}(W))) = 2\frac{K^n}{\Omega_n} \mu(W) \tag{3.28}$$

If we assemble all of the inequalites we have proven, and write $r^{n-1} = \Omega_{n-1}^{-1} m_{n-1}(W)$, we have proven

$$\Lambda(t, f(F_k))^n \leq \frac{2^{n+1} K^n \Omega_{n-1}(m_1(F) + \varepsilon)^{n-1} \mu(W)}{\Omega_n m_{n-1}(W)} \tag{3.29}$$

As $r \to 0$, $\mu(W)/m_{n-1}(W) \to \mu'(y) < \infty$, by our choice of y at the beginning (this is of course equivalent to saying $\mu(B(y, r))$ is on the order of r^{n-1}). So if we first let $r \to 0$, then $\varepsilon \to 0$, and lastly $t \to 0$, we obtain

$$\Lambda(f(F_k))^n \leq C\mu'(y) m_1(F)^{n-1} \tag{3.30}$$

for a constant C. Letting $k \to \infty$, the compact sets F_k exhaust F, hence the compact sets $f(F_k)$ exhaust $f(F)$, and therefore

$$\Lambda(f(F))^n \leq C\mu'(y) m_1(F)^{n-1} \tag{3.31}$$

It is easy to see that this inequality for each compact subset F of $J \cap \mathrm{int}(Q)$ implies that $f|_J$ satisfies the hypotheses of Lemma 2.4. Therefore $f|_J$ is absolutely continuous. □

3.3. Finishing the proof of differentiability.

It remains only to show that if f is a quasiconformal homeomorphism, then $H(x) < \infty$ a.e., where H is defined in 3.4. Compared to the previous two theorems, this is easy.

Lemma 3.5. *Suppose that $f : U \to V$ is a quasiconformal homeomorphism. Then $H(x) < \infty$ a.e.*

Proof. Since f is a homeomorphism, we may consider the volume derivative ν'_f of f, defined in equation (2.7). ν'_f is locally integrable, hence finite a.e. We claim that for any $y \in U$, if $\nu'_f(y) < \infty$, then $H(y) < \infty$. Find positive numbers K, δ such that for $0 < r < \delta$, $D_O(y, r) \leq KD_I(y, r)$. By the Jordan curve argument in the proof of Theorem 3.4, $D_I(y,r)^n \leq \Omega_n^{-1} m(f(B(y,r)))$. Combining these inequalities yields an inequality which is equivalent to

$$\frac{D_O(y,r)^n}{r^n} \leq K^n \frac{m(f(B(y,r)))}{m(B(y,r))} \tag{3.32}$$

Letting $r \to 0$, we obtain

$$\limsup_{r\to 0} \frac{D_O(y,r)^n}{r^n} \leq K^n \nu_f'(y) < \infty \tag{3.33}$$

Since $D_O(y,r) \geq |f(y+v) - f(y)|$ for any v with $|v| = r$, this implies that $H(y) < \infty$. Hence $H(y) < \infty$ for a.e. $y \in U$. □

Combining Theorem 3.1, Theorem 3.4, and Lemma 3.5, we obtain

Theorem 3.6. *Let f be a quasiconformal homeomorphism. Then f is differentiable a.e.*