ERGODICITY of Geodesic flow & $\pi_1 \curvearrowright S^{n-1} \times S^{n-1}$.

Def: X, μ measure space, $G \curvearrowright X$. (Group or semigroup)

μ is ~~G~~-invariant (G measure preserving) if $\mu(g^{-1}(B)) = \mu(B)$ for all B mble

quasi-invariant if $g^{-1}(B)$ null set $\Leftrightarrow$ B null set.

Def: $G \curvearrowright (X, \mu)$ (quasi-) invariant.

Action is ergodic if any invariant (mble) set is full or null measure

$\Leftrightarrow$ any measurable $f : X \to \mathbb{R}$ s.t. $f \circ g = f \ \forall g \in G$ is a.e. constant.

(Proof of $\Leftrightarrow$: • Not ergodic for 1) $\Rightarrow \exists$ invariant A non full/null, let $f = \mathbb{1}_A$.

• ergodic for 1). Given $f : X \to \mathbb{R}$ invariant, for all $q \in \mathbb{Q}$

$\{x \mid f(x) < q\}$ is invariant mble so full or null measure.

let $r = \sup_{q \in \mathbb{Q}} \{ q \mid f(x) < q \text{ on null set} \}$, show $f \equiv r$ a.e.)

This also shows: $\Leftrightarrow$ any bounded $f : X \to \mathbb{R}$ is a.e. constant.

Examples:

1. Irrational rotation of $S^1 = \mathbb{R}/\mathbb{Z}$

$x \longmapsto x + \alpha \pmod{\mathbb{Z}}$ where $\alpha \notin \mathbb{Q}$. μ = Lebesgue measure

$G \cong \mathbb{Z}$ generated by T

Pf: take bounded $f : S^1 \to \mathbb{R}$, Fourier series for f $\sum_{-\infty}^{\infty} a_n e^{2n\pi i x}$ L^2-converges to f, $f \circ R_\alpha$ is limit of series $\sum a_n e^{2n\pi i(x+\alpha)}$. Uniqueness of Fourier coeff $\Rightarrow a_n = a_n e^{2n\pi i \alpha}$ $\Rightarrow a_n = 0 \ \forall n \neq 0$ $\Rightarrow f$ constant.

2. (Semigroup) action of $T_2 : S^1 \to S^1$, $x \mapsto 2x$. Lebesgue measure. Note - invariance!

3. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \curvearrowright \mathbb{R}^2/\mathbb{Z}^2 = T^2$ (or any $A \curvearrowright \mathbb{R}^d/\mathbb{Z}^d$, $A \in SL_d\mathbb{Z}$ where A has no eigenvalue a root of unity)

(Lebesgue measure/area) preserved since $\det = 1$

4. OUR GOAL: Geodesic flow on UTM

($\mathbb{R}$ - action) M finite vol. hyperbolic mfld.

Birkhoff Ergodic theorem: "Time average = Space average"

Suppose $\varphi_t \curvearrowright X, \mu$ measure preserv. $\mu(X) = 1$.

$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\varphi_t(x))\,dt$ exists for a.e. x, and $f^+ \in L^1_\mu$

$= f^+(x)$ "average of f along orbit in positive time"

If furthermore φ_t is ergodic, then ② $\underbrace{\int_X f^+(x)\,d\mu}$ $= \int_X f(x)\,d\mu$. ① f^+ is φ_t-invariant, so a.e. constant $= \int_X f(x)$.

measure of A. (space)

Note: if $f = \mathbb{1}_A$, $f^+(x)$ is average time trajectory from x spends in A

→ "average of average time in A"

(Same works for analogous f^-, and $f^- \underset{a.e.}{=} f^+$ even w/o. ergodicity) [Proof: standard]

Analogous version for $\mathbb{Z}$ action "$\langle T\rangle$" (or $\mathbb{N}$ semi-grp) replacing LHS integral w/ $\lim_{n\to\infty} \frac{1}{n}\sum f(T^n(x))$

(see proof of both on page 5-6)

Cor (Converse): φ_t ergodic on X $\iff$ $\forall f \in L^1_\mu$,

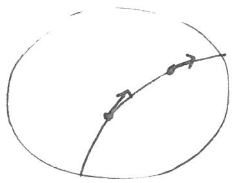
$$\int_X f(x)\,d\mu = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\varphi_t(x))\,dt \quad \text{a.e.}$$

(suffices to check on a dense subset of such L^1 functions)

Pf: $(\Leftarrow)$ if RHS holds,

given φ-invariant f, (bounded say so in L^1) $f(\varphi_t) = f$ so $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\varphi_t(x))$ $= f(x)$ so constant a.e.

Geodesic flow on $UT\,\mathbb{H}^n$:

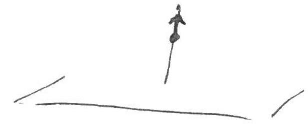


$$UT\,\mathbb{H}^n \underset{\text{diffeo}}{\cong} \left(S^{n-1}\times S^{n-1} \setminus \Delta\right) \times \mathbb{R}$$

endpts of oriented geodesic; parametize geodesic by setting 0 as midpt of circular arc in Ball model, parametrize by arc length.

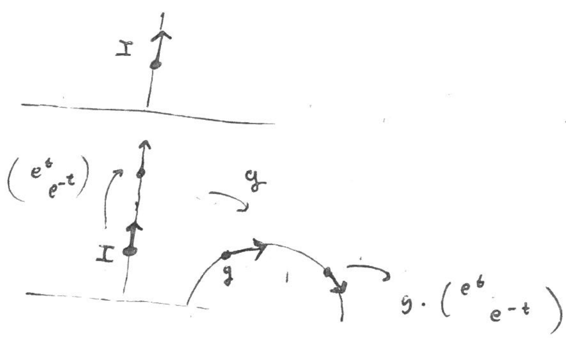
In these coords, flow is: $\phi_t(a,b,x) = (a,b,x+t)$

More intrinsically: $UT\,\mathbb{H}^n \cong$ ~~Isom($\mathbb{H}^n$)~~ $SO(n,1)\big/$ stabilizer of vector for $SO(n,1) \curvearrowright UT\,\mathbb{H}^n$ $= SO(n,1)\big/SO(n-1)$



many!)
Invariant measures:
- arc length on closed geodesic
- [calculus exercise] $\underbrace{d\mathrm{vol}}_{\text{Riemann volume.}} \times \underbrace{d\theta}_{\text{angle measure on unit sphere in } T_x}$

In $SO(2,1) \cong PSL(2,\mathbb{R})$
$\overset{\shortparallel}{UT(\mathbb{H}^2)}$



Flow is right-mult. by $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$

$\Rightarrow$ Preserves right-invariant Haar measure.

(similar works for $SO(n,1)$ but 1-parameter subgp uglier to write).

Haar is abs. continuous wrt. Lebesgue on $\left(S^{n-1}\times S^{n-1} - \Delta\right)\times\mathbb{R}$, so can think of this for full/null sets.

Remark: Many other measures are ~~also~~ preserved (e.g. arc length on a single closed geodesic)

Proof (idea) of ergodicity: (Hopf)

Let M have finite vol. $f \in L^1_\mu$. By previous remark/cor. suffices to just check for f compactly supported continuous functions on M (these are dense).

Lift f to $\tilde{f}$ on $UT(\mathbb{H}^n)$, cpt supp. in each fund. domain.

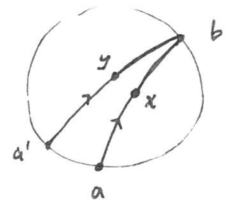
let $f^+(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \bar{f}\cdot(\varphi_t(x))\, dt$ a.e. defined by Birkhoff.

This depends only on geodesic through x, so a function of endpoints
$F(a,b) : S^{n-1}\times S^{n-1} - \Delta \to \mathbb{R}$

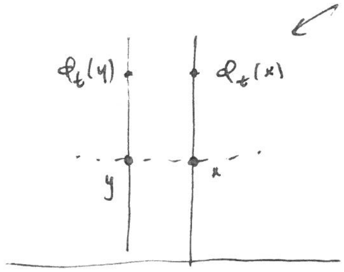
(not on starting point)

<u>Claim</u>: F is independent of a.

<u>Proof</u>:



Choose x, y s.t. distance between $\varphi_t(x)$ and $\varphi_t(y) \to 0$ as $t\to\infty$ (On the same horocycle)



f compactly supported continuous $\Rightarrow$ unif. continuous

$\Rightarrow f^+(x) = f^+(y)$ ✓

Similarly, analogous $f^-(x)$ is indep. of b.

<u>Birkhoff</u>: $f^+ = f^-$ a.e. so F is a.e. indep. of a <u>and</u> b hence constant.

<u>Ergodicity of $\pi_1 \curvearrowright S^{n-1}\times S^{n-1}$.</u>

Suppose A some π_1-invariant set

then $A\times\mathbb{R}$ is π_1-invariant, and flow-invariant so $A\times\mathbb{R}/\pi_1$ is flow-invariant

$\Rightarrow$ full or null measure in $UT(M)$

In 1st case

$\Rightarrow A\times\mathbb{R}$ null measure in $UT(\mathbb{H}^n)$

$\Rightarrow A$ null in $S^{n-1}\times S^{n-1}$

(2nd case: look at complement, a null set)

☒

Proof of Birkhoff for $\mathbb{Z}$-action:

Thm: (Positive orbits). X, μ measure space (not nec. finite).
$\varphi: X \to X$ measure presery, $f: X \to \mathbb{R}$ integrable, $\int_X f \, d\mu > 0$

Then $\exists$ orbit $x, \varphi(x), \varphi^2(x), \ldots$ such that $\sum_{i=0}^{n-1} f \cdot \varphi^i(x) > 0$.

Proof has tricks! Take as black box & use to prove Birkhoff.

Given $\varphi: X \to X$ measure presery, $\mu(X) = 1$, $f: X \to \mathbb{R}$

let $\overline{f}(x) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))$, $\underline{f}(x) = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x))$.

Then follows from defn' that $\overline{f} \circ \varphi(x) = \overline{f}(x)$
$\underline{f}\varphi(x) = \underline{f}(x)$

Suppose first $\overline{f}$ was bounded, so integrable.

Claim: $\int \overline{f} \leq \int f$

Proof: If not, $\exists \varepsilon > 0$ s.t. $\int \overline{f} > \int (f + \varepsilon)$, so $\int (\overline{f} - (f + \varepsilon)) > 0$

Pos. orbit thm $\Rightarrow$ $\exists$ orbit where $\sum_{i=0}^{n-1} (\overline{f} - (f + \varepsilon)) \varphi^i(x) > 0$

i.e. $\sum_{i=0}^{n-1} \overline{f} \varphi^i(x) > \sum_{i=0}^{n-1} f \varphi^i(x) + n \cdot \varepsilon$

$\qquad \| $

$n \cdot \overline{f}(x)$ by invariance

divide by $\frac{1}{n}$ take limit $\Rightarrow$

$\overline{f}(x) > \overline{f}(x) + \varepsilon$ $\lightning$

Similarly $\int \underline{f} \geq \int f$, so we have

$$\int \overline{f} \leq \int f \leq \int \underline{f} \underset{\text{by defn'}}{\leq} \int \overline{f}$$

so lim inf & lim sup agree a.e.
(f)

If $\bar{f}$ not bounded, consider φ-invariant set $X_n := \{x \mid |\bar{f}(x)| \le n\}$ and apply same argument there to conclude that $\int_{X_n} \bar{f} = \int_{X_n} \underline{f}$ so these agree a.e. on X_n.

True for all $n \Rightarrow \underline{f} = \bar{f}$ a.e.

Invertible φ: If φ^{-1} is a function, then could also take $f^{-}(x) := \sum_{i=0}^{n-1} f \cdot \varphi^{-i}(x)$, the same argument shows it is well defined.

Claim: $f^{-}(x) = f^{+}(x)$ for a.e. x.

Proof: let $N = \{x \mid f^{-}(x) > f^{+}(x)\}$. This is a φ-invariant set.

Then $\int_N f^{-} = \int_N f = \int_N f^{+}$ so N must have measure zero!

Proof for $\mathbb{Z}$-action implies proof for flow:

Suppose we know Birkhoff for $\mathbb{Z}$-actions.

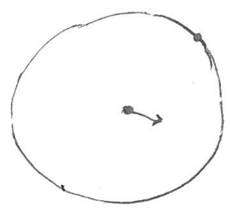
Let φ_t be a measure-preserving flow. Then for $\varphi := \varphi_1$, have $\varphi^n = \varphi_n$.

Given φ-invariant $f : X \to \mathbb{R}$, consider $\hat{f} := \int_0^1 f \circ \varphi_t(x)\, dt$.

Then $\int_0^N f \circ \varphi_t(x)\, dt = \sum_{i=0}^{N} \hat{f} \circ \varphi_t(x)$ so can apply discrete version to $\hat{f}$ to prove result for f.

Alternative short ending to proof of Mostow: (using a harder ergodicity theorem)

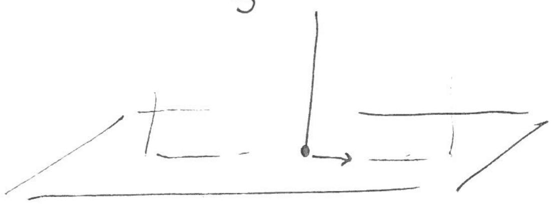
Use fact that boundary map is differentiable a.e.



S^{n-1}

$\mathrm{Isom}(\mathbb{H}^n) = SO(n,1) \curvearrowright S^{n-1}$ by conformal transformations

$\curvearrowright PT(S^{n-1})$.



$\partial \mathbb{H}^n$

Stabilizer of a vector contains $\mathbb{R}$-subgroup of dilations $z \mapsto \lambda z$ in upper half plane so is some noncompact, closed subgroup H.

Thm (Moore ergodicity) G noncompact, simple Lie group w/ finite center. $H < G$ closed, noncompact., Γ discrete, G/Γ finite volume.

Then $\Gamma \curvearrowright G/H$ is ergodic

(wrt. right Haar, measure class is preserved).

End of proof: (Bourdon)

Define $h : PT(S^{n-1}) \to \mathbb{R}$

via: $v \in T_z S^{n-1}$ $\quad [v] \in PT$

$$h([v]) = \frac{\| D_z F(v) \|}{\|v\| \cdot \|D_z F\|}$$

"how much v is stretched compared to max. stretch"

$\|D_z F\|$ ← "λ_1" from previous lecture.

This is Γ-invariant: $h(\gamma \cdot v) = \dfrac{\cancel{D_z F(\gamma v)} \, \| D_z \gamma(v) \cdot D_{\gamma z} F(\gamma \cdot v) \|}{\ldots}$

$\Gamma = \pi_1 M_1$

$\|D_z \gamma \cdot v\| = \lambda \|v\|$ $\quad \| \; = \lambda$.

conformal, so stretch by λ