Math 130 Homework 4

Reading:

- Hartshorne, sections 6, 7 and 8. (they look like chapters, but they are sections in chapter 2)
- Optional, if you're interested: History of the Jordan curve theorem at `en.wikipedia.org/wiki/Jordan_curve_theorem`

1. (The rational cartesian plane). Consider a geometry where the points are ordered pairs (a, b) of *rational* numbers, and the lines are all the subsets of the form
$$\{(x_0, y_0) + t(x_1, y_1) : t \in \mathbb{Q}\}$$
where (x_0, y_0) and (x_1, y_1) are fixed points in $\mathbb{Q} \times \mathbb{Q}$, $(x_1, y_1) \neq (0, 0)$.
Prove that this satisfies axioms I1 – I3. Which of the betweenness axioms are satisfied? (use the same notion of betweenness (for $\mathbb{R}^2$) discussed in Thursday's lecture – it's also the same as the one in Hartshorne)

2. Can you replace $\mathbb{Q}$ with $\mathbb{Q}(\sqrt{2})$ in Problem 1? What about any other subfield of $\mathbb{R}$?
(To answer this question, you need to give a clear explanation of whether your previous work used special properties of $\mathbb{Q}$, or if every property of $\mathbb{Q}$ that you used also works for other subfields of $\mathbb{R}$. You don't need to re-prove everything)

3. Do the following problems from Hartshorne chapter 6:
6.3a) (also draw a pictorial representation of what such a plane looks like, like we did for the 5-point plane in class)
6.10 (hint: induction)

4. Verify that the *real projective plane* that we introduced in class satisfies the axioms P1–P4 for a projective plane given in Hartshorne problem 6.3.
Recall that the points are $\mathbb{R}^2 \cup [0, \pi)$ and the lines are all of the form
$\{(x, y), \theta : y = mx + b \text{ and } \theta = tan(b)\}$ or $\{(a, y), \pi/2\}$ (vertical lines), or $\{\theta : 0 \leq \theta < \pi\}$ (all the slopes together).

5. Define a reasonable notion of betweenness for the real projective plane that extends the notion of betweenness in $\mathbb{R}^2$. Does this satisfy axioms B1 to B4?
(you may have different answers from each other, as I am giving you some freedom of how to define "between")

6. Read (and learn) the proof of *Plane Separation*, proposition 7.1 in Hartshorne. (nothing to hand in).

7. Do the following problems from Hartshorne chapter 7:
7.1, 7.4, 7.9, 7.10