Math 130 Homework 8

Reading:

- Stillwell, chapter 7. We will not discuss 7.8, nor the "24 cell" at the end of 7.7
- More than you ever wanted to know about quaternions and rotation: `https://en.wikipedia.org/wiki/Quaternions_and_spatial_rotation`. But you might be especially interested in the section on practical advantages

1. Show that every isometry of $\mathbb{R}^2$ can be expressed as the composition of three reflections. (you may wish to look at S4P, section 3.7, but you should give your own proof. In particular, you can use things we did in class.)
2. Do the following problems from S4P: 7.1.1 (it is ok to use anything we did in class!), 7.1.3.
3. Do the following problems from S4P: 7.2.2 - 7.2.6
4. The *affine transformations of* $\mathbb{R}^2$ is the group of functions of the form $f(\vec{x}) = M\vec{x}+\vec{w}$, where $M \in \mathrm{GL}_2(\mathbb{R})$ and $\vec{w} \in \mathbb{R}^2$. (Compare with the affine transformations of $\mathbb{R}$, i.e. $f(x) = ax + b$, with $a \in \mathrm{GL}_1(\mathbb{R})$, $b \in \mathbb{R}$.)
   (a) If $f_1(\vec{x}) = M_1\vec{x} + \vec{w}_1$ and $f_2(\vec{x}) = M_2\vec{x} + \vec{w}_2$, what is $f_1 \circ f_2$?
   (b) If f_1 is as above, what is its inverse?
   (c) Give examples of two concepts that are invariant under affine transformations
5. Following Chapter 7.6 of S4P, let $\mathbf{q} = \cos(\theta/2) + \mathbf{i}\sin(\theta/2)$.
   (a) Write $\mathbf{q}$ as a 2×2 matrix.
   (b) Verify that the isometry $\mathbf{p} \mapsto \mathbf{qpq}^{-1}$ of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$-space leaves the $\mathbf{i}$-axis fixed and rotates the $(\mathbf{j}, \mathbf{k})$-plane through an angle θ.
6. (based on S4P 7.7) Consider an *octahedron* in $\mathbb{R}^3$ with vertices $\pm(1, 0, 0)$, $\pm(0, 1, 0)$ and $\pm(0, 0, 1)$. How many elements are in the group of rotations of the octahedron? What quaternions make up the group of rotations of the octahedron?
7. Optional challenge (not to hand in): Why is there no orientation-preserving isometry of the sphere that does not fix any points? Can you prove this by imitating our strategies from in class? Or do you need to use a "three reflections theorem"? You might find it helpful to do the exercises at the end of section 7.4 as a start.