Math 130 Worksheet 5: Rotations

Every rotation of $\mathbb{R}^2$ about the origin has the following matrix form

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Alternatively, one can think of $\mathbb{R}^2$ as the *complex plane*. In this case, rotation corresponds to multiplication by a complex number $a + bi$ where $a^2 + b^2 = 1$.

Exercise: What is the result of multiplying $x + iy$ by i?
Why does this correspond to rotating the complex plane by 90°?

In three dimensions, the situation is a bit more complicated.
Given a unit vector $u = (u_x, u_y, u_z)$, where $u_x^2 + u_y^2 + u_z^2 = 1$, the matrix for a rotation by an angle of θ about an axis in the direction of u is:

$$R = \begin{pmatrix} \cos\theta + u_x^2\,(1-\cos\theta) & u_x u_y\,(1-\cos\theta) - u_z \sin\theta & u_x u_z\,(1-\cos\theta) + u_y \sin\theta \\ u_y u_x\,(1-\cos\theta) + u_z \sin\theta & \cos\theta + u_y^2\,(1-\cos\theta) & u_y u_z\,(1-\cos\theta) - u_x \sin\theta \\ u_z u_x\,(1-\cos\theta) - u_y \sin\theta & u_z u_y\,(1-\cos\theta) + u_x \sin\theta & \cos\theta + u_z^2\,(1-\cos\theta) \end{pmatrix}.$$

(Using some linear algebra, it is possible to prove that these are exactly the *orthogonal matrices of determinant 1*, meaning that $M^T M = I$ and $\det(M) = 1$).

Exercise: What is the matrix for rotating by θ degrees about the *vertical axis*, $u = (0, 0, 1)$?

What's next: Our goal today is to see a different way to describe rotations of $\mathbb{R}^3$, analogous to the complex numbers for rotating $\mathbb{R}^2$.

(Optional exercise if you have extra time:) What complex number do you need to multiply by to get rotation by 45°?