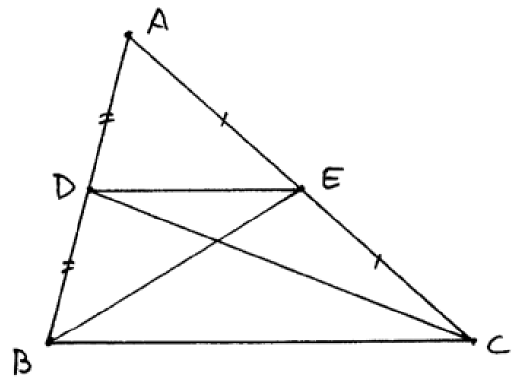
3.11 (Campanus). Use the theory of content to show that the line DE joining the midpoint of two sides of a triangle is parallel to the third side. (*Hint*: Draw BE and DC. Show that the triangles BDC and BEC have the same content and then apply (I.39).)



4 Construction of the Regular Pentagon

One of the most beautiful results in all of Euclid's *Elements* is the construction of a regular pentagon inscribed in a circle (IV.11). The proof of this construction makes use of all the geometry he has developed so far, so that one could say that to understand fully this single result is tantamount to understanding all of the first four books of Euclid's geometry. It also raises questions of exposition that are central to our modern examination of Euclid's methods. For example, why does Euclid use the theory of area in proving a result about the sides of a polygon?

In this section we will present Euclid's construction of the regular pentagon, and begin discussing the issues raised by its proof. Later (see (13.4), Exercise 20.10, (29.1)), we will give other proofs using similar triangles or the complex numbers. Euclid's original geometric proof must be regarded as a tour de force of classical geometry. It depends on the theory of area, which we will discuss in more detail in Section 22. So this section can be regarded as a taste of things to come: a first meeting with one of the deeper topics that is central to Euclid's geometry.

The key point of the construction of the pentagon is the following problem.

Problem 4.1
To construct an isosceles triangle whose base angles are equal to twice the vertex angle.

Construction ((II.11), (IV.10))
Let A, B be two points chosen at random.

1. Draw line AB.

Next, construct a perpendicular to AB at A, as follows:

2. Circle AB, get C.
3. Circle BC.
4. Circle CB, get D.
5. Line AD, get E.

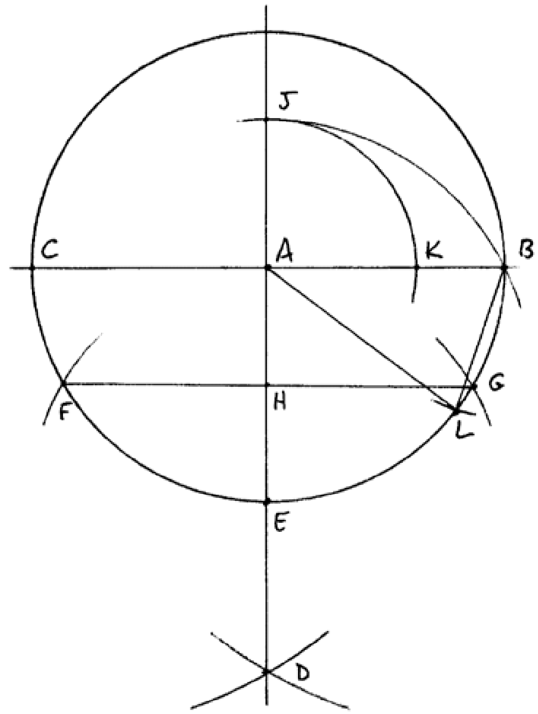
Next, we bisect AE as follows

6. Circle EA, get F, G.
7. Line FG, get H.

Now comes the unusual part of the construction:

8. Circle HB, get J.
9. Circle AJ, get K.
10. Circle center B, radius AK, get L.
11. Line AL.
12. Line BL.

Then ΔABL is the required triangle. The angles at B and at L will be equal to twice the angle at A.

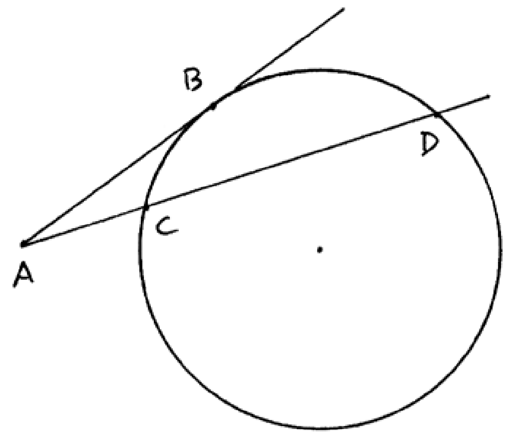


Proof From a modern point of view, it would seem that some theory of quadratic equations is essential for the proof. Euclid did not have any algebra available to him, but he was able to deal with quantities essentially equivalent to quadratic expressions via the theory of area. We can think of a rectangle as representing the product of its sides, or a square as the square of its side. These areas, without even assigning a numerical value to them, can be manipulated by cutting up and adding or subtracting congruent pieces. In this way Euclid establishes a "geometrical algebra" for manipulating these quantities (always by geometrical methods), which acts as a substitute for our modern algebraic methods.

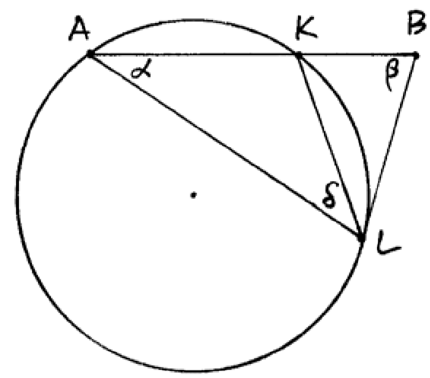
Let us then trace the steps by which Euclid proves (IV.10), which is the key point in the construction of the regular pentagon. In Book I, especially (I.35)–(I.47) he discusses the areas of triangles and parallelograms, leading up to the famous Pythagorean theorem (I.47), which is stated in terms of area: The square built on the hypotenuse of a right triangle has area equal to the combined areas of the squares on the two sides. The theorem is proved by cutting these areas into triangles, and proving equality of areas using the cutting and pasting methods just developed. Here area is understood in the sense of content—cf. Section 3.

Book II contains a number of results of geometrical algebra, as described above, all stated and proved geometrically in terms of areas. In particular, (II.5), (II.6), and (II.11) are used in the proof of (IV.10). Note that (II.11), which is sometimes called the division of a segment in extreme and mean ratio, states that the interval AB is divided by a point K (in our notation (4.1) above) such that the rectangle formed by BK and AB has area equal to the square on AK. In this way the property of extreme and mean ratio is expressed using area.

From Book III we need (III.36) and its converse (III.37). Proposition (III.36) says that if a point A lies outside a circle, and if AB is tangent to the circle at B, and if ACD cuts the circle at C and D, then the rectangle formed by AC and AD has area equal to the square on AB. This result is proved by several applications of (II.6) and (I.47).



Now Euclid can prove (IV.10) by a brilliant application of (III.37). Let A, K, B, L be as in the construction (4.1) above. Then by (II.11), the rectangle with sides BK and BA has area equal to the square on AK. Since BL was constructed equal to AK, this is also equal to the square on BL.



Now consider the circle passing through the three points A, K, L. Since the rectangle on BK and BA is equal to the square on BL, it follows that BL is tangent to this circle (III.37)!

Hence the angle $\angle BLK$ formed by the tangent BL and the line LK is equal to the angle α at A, which subtends the same arc (III.32). Let $\angle KLA = \delta$. Then $\angle BKL$ is an exterior angle to the triangle ΔAKL, so $\angle BKL = \alpha + \delta$ (I.32). But $\angle BLK = \alpha$, so $\alpha + \delta = \angle BLA$, and this angle is β because ΔABL is isosceles. Hence $\angle BKL = \beta$. Now it follows that ΔBKL is isosceles, so $KL = BL = AK$. Hence ΔAKL is also isosceles, so $\delta = \alpha$. Now $\beta = \angle BLA = 2\alpha$ as required.

Once we have the isosceles triangle constructed in (4.1), the construction of the pentagon follows naturally. The idea is to inscribe in the circle a triangle equiangular with the given triangle, and then to bisect its two base angles.

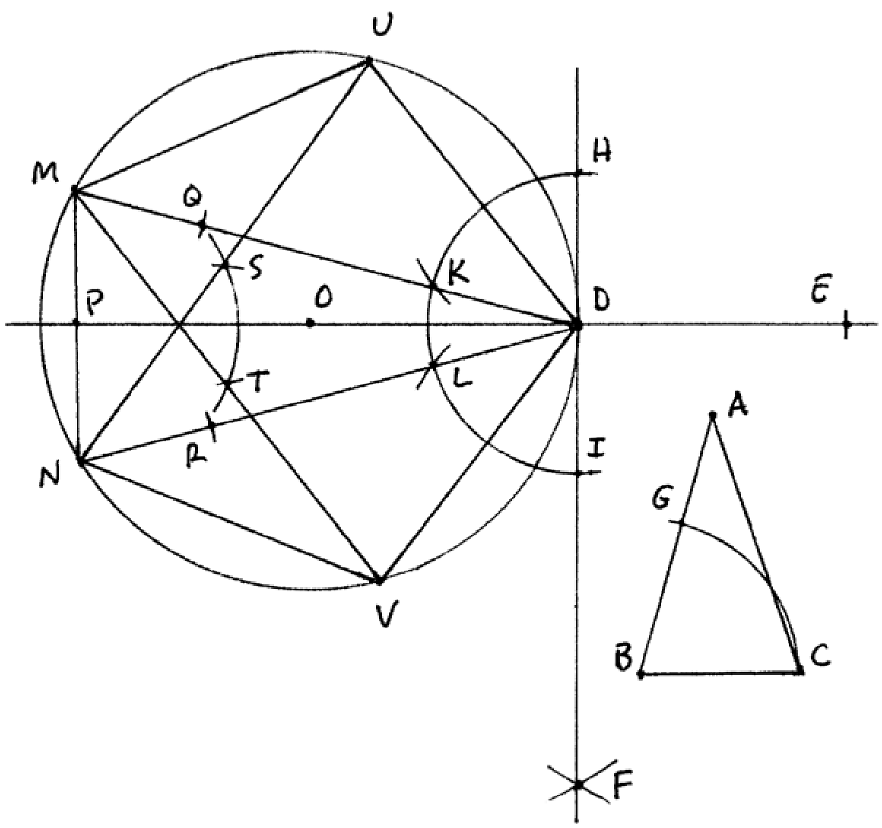
Problem 4.2
Given an isosceles triangle whose base angles are equal to twice its vertex angle, and given a circle with its center, to construct a regular pentagon inscribed in the circle.

Construction ((IV.2) and (IV.11))
Let ΔABC be the given triangle and let O be the center of the given circle. The first part of the construction is to obtain a tangent line to the circle. Let D be any point on the circle.

1. Line OD.
2. Circle DO, get E.
3. Circle EO.
4. Circle OE, get F.
5. Line DF.

Then DF will be a tangent line. Next, we reproduce the angle β from the base of the isosceles triangle at D, on both sides.

6. Circle BC, get G.
7. Circle at D with radius equal to BC, get H, I.
8. Circle center H, radius CG, get K.
9. Circle center I, radius CG, get L.
10. Line DK, get M.
11. Line DL, get N.
12. Line MN.



Then ΔDMN is a triangle inscribed in the circle, equiangular with ΔABC. Next we bisect the angles at M, N. Let P be the intersection of MN with DO.

13. Circle MP, get Q.
14. Circle NP, get R.
15. Circle PR.
16. Circle RP, get S.
17. Circle QP, get T.
18. Line NS, get U.
19. Line MT, get V.

Then D, M, N, U, V will be the vertices of the pentagon.

20. Line DU.
21. Line UM.
22. Line DV.
23. Line VN.

Then $DUMNV$ is the required pentagon.

Proof We follow the geometric proof given by Euclid. First of all, the line DF is constructed perpendicular to a diameter of the circle, so it is a tangent line to the circle (III.16). Next, the triangles ΔDHK and ΔDLI are constructed so that their three sides are equal to the three sides of ΔBCG. Hence by (SSS) = (I.8), it follows that $\angle KDH$ and $\angle LDI$ are both equal to the angle β of the triangle ΔABC at B. From there it follows that the angles of ΔDMN at M and N are both equal to β, because they subtend the same arcs cut off by the tangent line and the angles β just constructed (III.32). Since the sum of the three angles of a triangle is constant $= 180°$ (I.32), it follows that the triangle ΔDMN is equiangular with the triangle ΔABC. In particular, if α is the angle at D, then $\beta = 2\alpha$.

The points U, V are constructed by taking the angle bisectors of ΔDMN at M and N. Since the angles at M and N are β, their halves are equal to α. Thus the arcs DU, UM subtend angles α at N; the arc MN subtends an angle α at D; and the arcs DV, VN subtend angles α at M. Hence these five arcs are all equal (III.26), and the line segments on them are also equal. So we have constructed an equilateral pentagon inscribed in the circle. The angle subtended by each side at the center of the circle will be $2\alpha = \beta$. It follows that the angles of the pentagon are also equal, so the pentagon is *regular* in the sense that its sides are all equal and its angles are all equal.

This completes the presentation of Euclid's construction of the pentagon. As usual, his method is adapted to economy of proof, not economy of steps used. The whole construction, as we have presented it here, takes $12 + 23 = 35$ steps. By collapsing separate parts of the construction, in particular, by constructing the triangle of (4.1) on a radius of the given circle, one can make a construction with fewer than half as many steps (cf. (4.3)). Note also that Euclid's construction of the points U, V by bisecting the angles at M, N makes possible his elegant proof that the five sides of the pentagon are equal. However, in retrospect we see that MN is actually one side of the pentagon, so U and V could have been constructed in a single step by a circle with center D and radius MN.

If there is such a thing as beauty in a mathematical proof, I believe that this proof of Euclid's for the construction of the regular pentagon sets the standard for a beautiful proof. In the words of Edna St. Vincent Millay, "Euclid alone has looked on beauty bare."

Now let us use the ideas of Euclid's method to construct a pentagon in as few steps as possible.

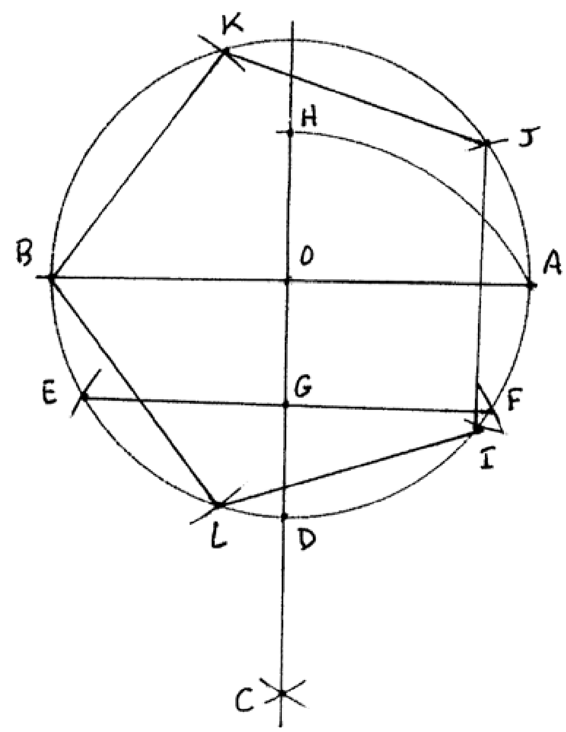
Problem 4.3

Given a circle with center O, construct a regular pentagon inscribed in the circle in as few steps as possible.

1. Draw any line through O. Get A, B.
2. Circle AB.
3. Circle BA, get C.
4. OC, get D.
5. Circle DO. Get E, F.
6. EF, get G.
7. Circle GA, get H.
8. Circle center A, radius OH, get I, J.
9. Circle center B, radius IJ, get K, L.

10–14. Draw BK, KJ, JI, IL, LB.

Then $BKJIL$ is the required pentagon.



Exercises

4.1 Read Euclid, Book IV.

4.2 Explain why the construction of (Problem 4.3) gives a regular pentagon.

4.3 Given a circle, but not given its center, construct an inscribed equilateral triangle in as few steps as possible (par = 7).