

The Continuum Hypothesis: how big is infinity?

When a set theorist talks about the *cardinality* of a set, she means the *size* of the set. For finite sets, this is easy. The set $\{1, 15, 9, 12\}$ has four elements, so has cardinality four. For infinite sets, such as \mathbb{N} , the set of even numbers, or \mathbb{R} , we have to be more careful. It turns out that infinity comes in different sizes, but the question of which infinite sets are “bigger” is a subtle one.

To understand how infinite sets can have different cardinalities, we need a good definition of what it means for sets to be of the same size. Mathematicians say that two sets are the same size (or have the same cardinality) if you can match up each element of one of them with exactly one element of the other. Every element in each set needs a partner, and no element can have more than one partner from the other set. If you can write down a rule for doing this – a matching – then the sets have the same cardinality. On the contrary, if you can prove that there is no possible matching, then the cardinalities of the two sets must be different. For example, $\{1, 15, 9, 12\}$ has the same cardinality as $\{1, 2, 3, 4\}$ since you can pair 1 with 1, 2 with 15, 3 with 9 and 4 with 12 for a perfect match.

What about the natural numbers and the even numbers? Here is a rule for a perfect matching: pair every natural number n with the even number $2n$. Since this pairs up each element of \mathbb{N} with exactly one even number, and since each natural number gets a *different* even number, we’ve made a perfect match. So we say that \mathbb{N} and the set of even numbers have the same cardinality.

This might seem counterintuitive: the even numbers are a *subset* of \mathbb{N} , in fact, you might have said earlier that there are *twice* as many natural numbers as evens, since in between every two even numbers there is another natural number. But, as we saw with the story of the Hilbert Hotel, infinity is tricky. The only way to accurately compare the sizes of sets is by making a match.

Now for the real numbers. It turns out – and this is a beautiful and famous argument due to set theorist Georg Cantor – that there is no possible matching between the real numbers and the naturals. The set of all real numbers has a strictly larger size.¹ This prompts the question: are there any sets that are bigger than the set of natural numbers, but smaller than the set of reals? That question is called the *Continuum Problem* and the answer “no” (that there are no sets of an in-between size) is called the *Continuum hypothesis* or “CH” for short. This problem has had quite a bit of fame: when Hilbert made a public list of the top 24 problems in mathematics in the year 1900, resolving CH came first. To this day, the truth or falsehood of CH is not entirely decided.

The first part of the answer to the continuum problem was due to Kurt Gödel. In 1938 Gödel proved that it is impossible to disprove CH using the usual axioms for set theory. So CH could be true, or it could be unprovable. In 1963 Paul Cohen finally showed that it was in fact unprovable. Much like how we added “cancellation for \mathbb{Z} ” to our list of axioms for arithmetic, if mathematicians wanted the continuum hypothesis to be true, then they would have to add it to their list of axioms.

In a 2004 paper called “Recent progress on the continuum hypothesis”, mathematician Patrick Dehornoy writes:

It might be tempting to conclude that the Continuum Problem cannot be solved, and, therefore, is not a closed question, but, at least, has no interest, since every further effort to solve it is doomed to failure. This interpretation is erroneous. One may judge that studying the Continuum Problem is inopportune if one finds little interest in the objects it involves: complicated subsets of \mathbb{R} , well-orderings whose existence relies on the axiom of choice . But,

¹However, the set of *rational* numbers has the same cardinality as \mathbb{N} does!

if one does not dismiss the question a priori, one must see that Gödel's and Cohen's results did not close it, but, rather, opened it. As shows the huge amount of results accumulated in Set Theory in the past decades, the ZFC system [our standard set theory axioms] does not exhaust our intuition of sets, and the conclusion should not be that the Continuum Hypothesis is neither true nor false, but simply that the ZFC system is incomplete, and has to be completed.

In Dehornoy's opinion, CH is interesting because it prompts us to question whether CH or its opposite would make a good axiom. It opens the question of recognizing which new axiom system is "most relevant for describing the mathematical world". He argues that the best way for mathematicians to decide whether to adopt CH (or its opposite) as an axiom is to look at other candidates for new axioms that set theorists like. If the continuum hypothesis is consistent with these – if it doesn't contradict them – then we should seriously think about adopting it. Naturally, the best axioms to compare the continuum hypothesis with are other axioms that say things about different sizes of infinity.² These are the *large cardinal axioms*. In Dehornoy's words:

Several preliminary questions arise: What can be a good axiom? What can mean "solving a problem such as the Continuum Problem from additional axioms. We shall come back to these questions starting from the case of arithmetic. Various axioms possibly completing ZFC will be considered below. For the moment, let us simply mention the large cardinal axioms. Intuitively, they are the most natural axioms, and they play a central role. These axioms assert that higher order infinities exist, that go beyond smaller infinities in the way the infinite goes beyond the finite. They come from iterating the basic principle of Set Theory, which is precisely to postulate that infinite sets exist. One reason for the success of large cardinal axioms is their efficiency in deciding a number of statements that ZFC cannot prove. The important point here is that it seems reasonable to consider the large cardinal axioms as true, or, at least, to consider as plausible only those axioms A that are compatible with the existence of large cardinals in the sense that no large cardinal axiom contradicts A .

So should we believe the continuum hypothesis or not? In his paper Dehornoy argues not, based on some very recent work in set theory by W. H. Woodin at UC Berkeley. But the majority of mathematicians are not so sure. I think it's a fair bet that it will be years before the issue is decided for good. However, considering that CH has been a major mathematical problem since 1900, we're getting pretty close.

Additional recommended reading on infinity:

- A nice exposition of the Hilbert hotel:
<http://opinionator.blogs.nytimes.com/2010/05/09/the-hilbert-hotel/>
- Your textbook works through Cantor's proof that there are more real numbers than rationals in section 8.5
- David Foster Wallace has written a book called "Everything and More: A Compact History of Infinity". I haven't read it, but I'm willing to bet it's pretty good. Wallace was a brilliant scholar of Philosophy and English (but not math!), so his history of math book should be quite accessible to the non-mathematician.

²Yes, the hierarchy of sizes of infinity doesn't stop at the reals being bigger than the naturals. There are sets that are bigger than the reals, and ones even bigger than those, and then some more...