Exploring Hyperbolic Space

adapted from *The Institute for Figuring:* http://theiff.org/oexhibits/

In 1997 Cornell University mathematician Daina Taimina finally worked out how to make a physical model of hyperbolic space that allows us to feel, and to tactilely explore, the properties of this unique geometry. The method she used was crochet.

Dr Taimina's inspiration was based on a suggestion that had been put forward in the 1970's by the geometer William Thurston (also now at Cornell)[1]. Noting that one of the qualities of hyperbolic space is that as you move away from a point the space around it expands exponentially, Thurston designed a paper model made up of thin cresent-shaped annuli taped together.

But Thurston's model is difficult to make, hard to handle, and inherently fragile. Taimina intuited that the essence of this construction could be implemented with knitting or crochet simply by increasing the number of stitches in each row. As you increase, the surface naturally begins to ruffle and crenellate. Taimina, who grew up in Latvia with a childhood steeped in feminine handicrafts, immediately set about making a model. At first she tried knitting - and you can indeed knit hyperbolic surfaces - but the large number of stitches on the needles quickly becomes unmanageable and Taimina realized that crochet offered the better approach.

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[1] I believe that the paper model made out of triangles that you saw in class is one of Thurston's ideas.

The beauty of Taimina's method is that many of the intrinsic properties of hyperbolic space now become visible to the eye and can be directly experienced by playing with the models. Geodesics – or straight lines – on the hyperbolic surface can be sewn onto the crochet texture for easy examination. Through the yellow lines in the model below look curved, folding along them demonstrably produces a clean straight line.

Another aspect of hyperbolic space that can be experienced with Taimina's models is the properties of triangles. In school we learn that the angles of a triangle always sum to 180 degrees. That is true on a Euclidean plane, but it is not true on a sphere or on a hyperbolic plane. On a sphere, the interior angles of a triangle always sum to more than 180Ú (a fact the reader may verify for themselves by drawing on a beach ball or balloon). On a hyperbolic surface the angles of a triangle add to less than 180Ú. Moreover, the larger triangle the smaller the angles will be. Until finally when the triangle points are infinitely far apart - making the largest triangle possible - the angles will sum to zero degrees! This [approximately] Ideal Triangle and its angular oddity can be seen on the model below:

[Crochet model showing an Ideal Triangle, whose angles sum to zero degrees]

So, what exactly is hyperbolic space, and where did it come from?

To get to the heart of this enigmatic topic we must go back to Euclid and the original axioms of geometry[2]. Long regarded as the model of intellectual rigor, Euclidean geometry is based on five supposedly self-evident propositions, or axioms. The first three are mundane enough: they define a line segment, an infinite straight line, and a circle. The fourth is also uncontroversial and states simply that all right angles are equal – a proposition necessary to ensure that the space we are working in is essentially the same everywhere. The property of sameness at every point is the defining quality of a geometry – for mathematically speaking there are wilder and more unruly spaces.

Euclid's fifth postulate also sounds entirely sensible: it defines the conditions for "parallel" lines. But from the beginning mathematicians sensed that this apparently reasonable proposition needed further investigation.

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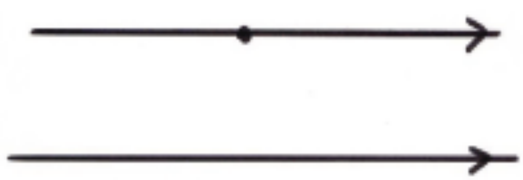
[2] Remember Euclid from your first reading from this course? It was a proof -- using his axioms -- that a certain construction gave an equilateral triangle.

There are several ways of describing this fifth, troublesome axiom, also known as the "parallel postulate." Euclid's own method is strange to modern eyes and mathematicians today prefer to use a construction discovered by the English mathematician William(TK) Playfair in the late nineteenth century. By Playfair's description we may understand parallel lines in the following way: Imagine that I draw a line, and then define a point P outside that line.

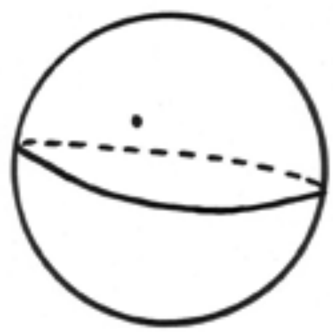
Now imagine that I draw various lines through the point P. What is the result? Euclid's fifth axiom says that there is only one line I can draw through P that will never meet the original line.



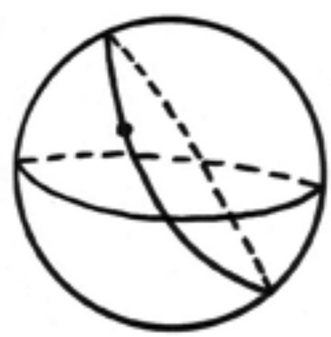
All other lines would slant with respect to the original line and eventually intersect it. We call the non-intersecting lines parallels and denote them by little arrows, indicating that they continue on indefinitely without meeting.

As stated above the parallel postulate seems intuitively correct. But there is an issue here that is suggested by the example of a sphere, whose surface forms another legitimate geometry. Again, we ask a question about the behavior of straight lines on the surface of a sphere. If we have a straight line drawn on a sphere and a point P outside this line (also on the sphere), then what happens when we try to draw other straight lines through the point P?

Immediately we are faced with the question what does it mean to say a “straight line” on a surface like this that is curved? Mathematically, a straight line on a flat surface may be generalized to the concept of a geodesic, a term which can be defined as the shortest path between any two points. On the surface of a sphere the shortest path between two points is always along a “great circle” - a circle that divides the sphere into two equal halves, like the earth’s Equator. Airlines use such geodesics when charting the paths of international flights, which often look curved on a flat map but are in fact “straight” in relation to the globe itself.



With respect to the sphere we notice immediately that the geodesics (straight lines) are not infinite as they are on a plane, but finite. They inevitably connect back up on themselves. Now returning to the initial question, we ask about the relationship between our original “straight line” and others we may draw through an external point P.



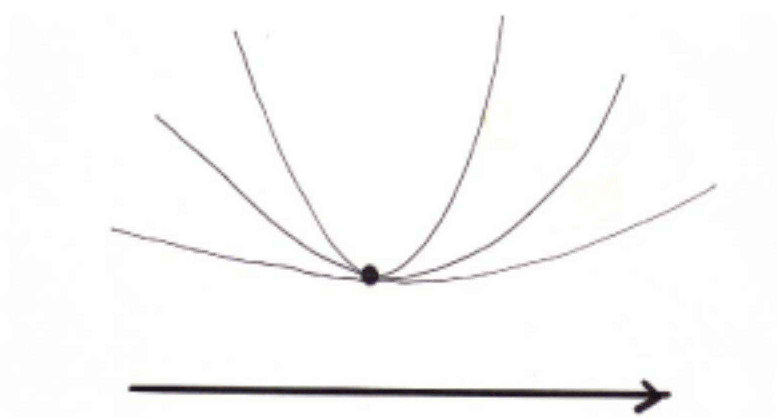
Any straight line through point P is also by definition a great circle - and all great circles on a sphere intersect one another. Thus on the surface of a sphere there are no straight lines through a point that do not meet the original line. Where on the plane there was one line that never met, now we have a geometry in which there are none.

How do we know that there may not be other options?

For two thousand years mathematicians sought to prove that the parallel postulate must be true, in the sense that there could never be more than one straight line though a point that did not meet a given line. The idea that this might not be true struck terror into their Euclidean hearts offending rational sensibilities and evoking a sense of moral outrage.

In order to prove the parallel postulate mathematicians resorted primarily to the method of reductio ad absurdum in which one begins by assuming the inverse of what ones hope to prove and then showing how this leads to a contradiction. That is, they assumed the parallel postulate was false and tried to show how that led inevitably to logical absurdity.
What they discovered was a host of absurdities – but, infuriatingly, no outright contradictions. Girolamo Saccheri, an eighteenth century Jesuit priest who devoted his life to the problem of parallels, went to his Maker a failure in his own eyes, unable to demonstrate, after Sisyphean effort, a single contradiction.

Finally, in the nineteenth century the effort to prove the parallel postulate exhausted itself, as mathematicians accepted the mounting evidence for the existence of a geometry based upon its absence. "I have created a new and different world," Janos Bolyai wrote to his father in 1823. In Russia, Nickolai Lobachevsky came to a similar insight. An alternative to Euclid, however disturbing, was logically undeniable. To put it into Playfair's terms, mathematicians were compelled to acknowledge that there exists a space in which given a line and an external point P, there are many lines that go through P yet do not meet the original line.



Instead of there being just one parallel to our original line, now there are many. Indeed there are infinitely many! Bizarre though it may seem, this situation gives rise to a consistent geometry, what came to be called, in homage to its abundant excess, hyperbolic space.

At this point the viewer may object that most of these lines do not look straight. That is merely because we are trying to see them from our limited Euclidean perspective. From the point of view of someone within the hyperbolic space, all these lines would be perfectly straight and infinitely long and none would meet the original line.

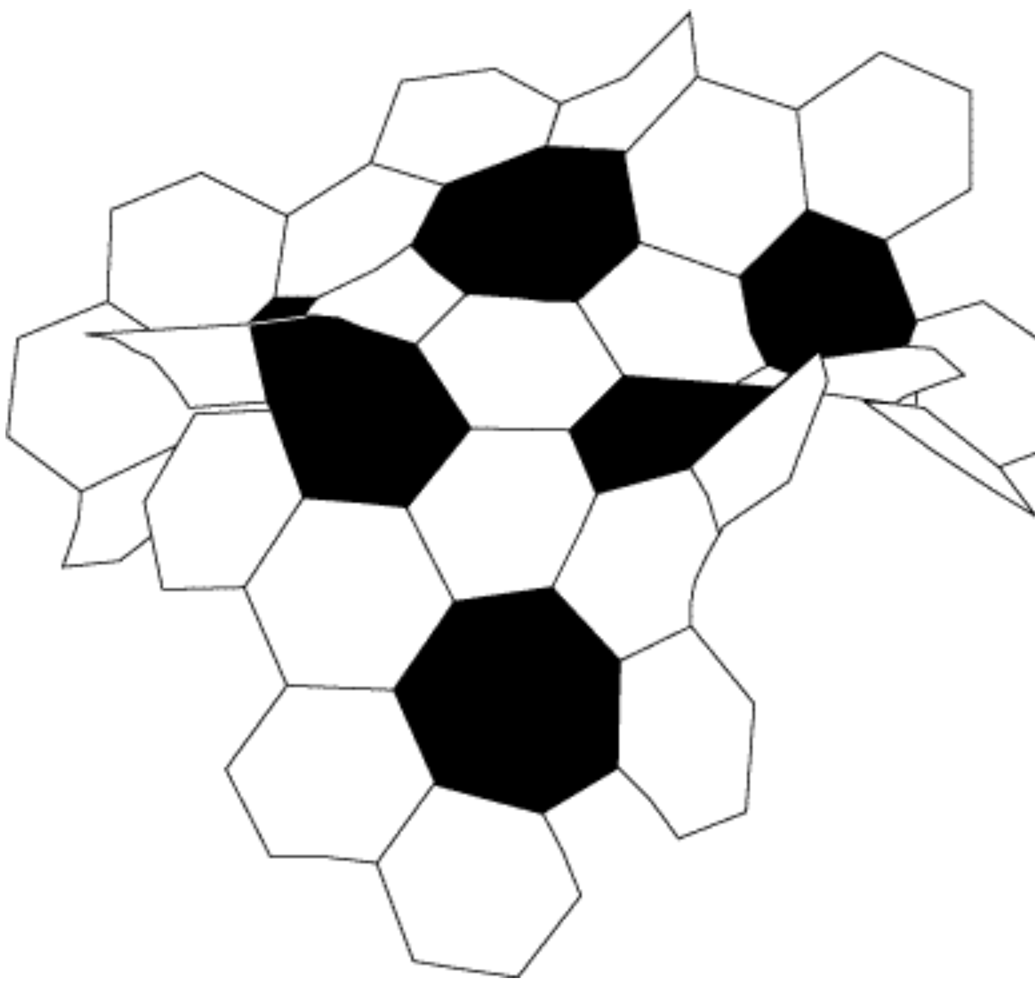
Below, the three lines through a point P are shown on one of the crocheted models:

There are other ways of visualizing the hyperbolic plane. One way is the "hyperbolic soccer ball". First think of a regular soccer ball: It is made up of hexagons and pentagons, with a series of white hexagons surrounding a smaller number of black pentagons.

Now think of the Euclidean plane (a regular flat plane). Here we can tile the surface with hexagons alone – the classic beehive pattern.

On the plane, every hexagon (which has six sides) is surrounded by six others and they all fit neatly together to exactly fill the space. To make a soccer ball, we replace some of the hexagons with pentagons (which only have five sides), thereby causing the now fewer number of hexagons to close up towards one another and wrap into a sphere.

In the hyperbolic version of this model we make the opposite move. Rather than replacing some of the hexagons with pentagons, we replace them with heptagons (which have seven sides). Now, instead of closing up, the surface opens out – for the heptagons add to, rather than subtract from the space, and what we get is an excess of surface.



The effect is similar to what we see in lettuce leaves and certain types of kelp where the vegetable surface expands outward from a modest start generating a ruffled effect. Mathematicians now understand that lettuces and kelps are in fact natural examples of hyperbolic geometry - which is also found in the anatomical frills of sea slugs, flatworms and nudibranches.

[a flatworm with hyperbolic ruffles!]

One problem with these models is their "ruffling" - hyperbolic space is two-dimensional, so it would be nice to have a genuinely two dimensional way of representing it -- if only so that it would be easier to illustrate on paper. [Like a map of the earth lets us represent the sphere on paper!]
One two-dimensional way of visualizing hyperbolic space was discovered by the great French mathematician Henri Poincaré [in fact, his model predates the ruffled models by over 50 years]. In the Poincaré disc model the entire hyperbolic space is depicted inside a circle.

In reality, hyperbolic space is infinitely large. Like the Euclidean plane it goes on forever. But in order for us to depict it within our Euclidean framework we have to make some compromises. The Poincaré compromise is to represent angles truly while distorting scale. In this diagram all the sides of all the triangular shaped areas are, in fact, of equal length.

In his book Science and Hypothesis (1901), Poincaré wrote of his model as an imaginary universe. To us, as observers of this bubble world, the inhabitants of the disc appear to shrink as they approach the boundary of the disc. They, however, see no such effect. As far as they are concerned, they live in a perfectly normal non-shrink space, albeit one that is not Euclidean. It is only we, confined to view them in a Euclidean framework, who see their dimensions behaving strangely.

The Poincaré disc model has entered the artistic lexicon through the work of the Dutch artist M.C. Escher, who was introduced to the concept by the great geometer Donald Coxeter. With his "Circle Limit" series of drawings, Escher explored the endless symmetries inherent in hyperbolic space: in "Circle Limit III," red, green, blue and yellow fish tessellate their world in a symphony of triangles and squares. In "Circle Limit IV" angels and demons disport themselves in a hyperbolic trinity, fluttering out from a central point to fill their space with hexagons and octagons.

In the playfulness of these images lies an elegant lesson: the excess of parallels opens up a far richer field for the tessellating spirit and the hyperbolic plane can be tiled in an infinite variety of ways. At the same time that Escher was propelled in these explorations by the formalities of hyperbolic geometry, he was also inspired by a visit to the Alhambra Palace in Spain, that apotheosis of the Arab world's unparalleled tiling tradition. If, as the Moors believed, repeated patterns connote the divine, we might conclude that Heaven itself would be a hyperbolic space.

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There is much more to hyperbolic space to explore. A great starting point is
the rest of the material at http://theiff.org/oexhibits/