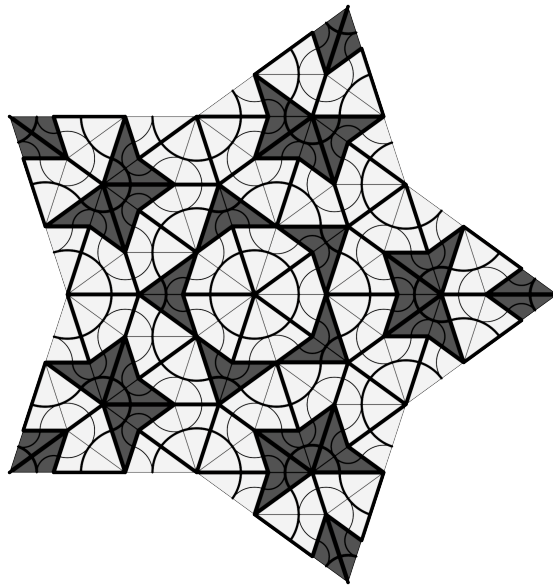


Penrose Tiles and Aperiodic Tessellations

Penrose tilings are a remarkable example of aperiodic, semi-regular tessellations¹. What follows is an excerpt from an article on Penrose tilings by Martin Gardner, from his book “Penrose Tiles to Trapdoor Ciphers”.

Gardner was the long-time author of a mathematics column in Scientific American magazine, as well as author of many books. He, more than anyone else, has made major contributions to making deep mathematics accessible to general audiences. His work demonstrates that, much like one need not be a Yeats scholar to appreciate and understand poetry, one need not be a professional mathematician to appreciate, understand, and find beauty and value in mathematics.

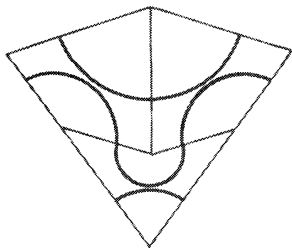
This short excerpt covers many of the ideas we talked briefly about in class and illustrates them with pictures and examples.



¹exercise: define all of the technical terms in that sentence!

CHAPTER 7

Penrose Tiling



At the end of a 1975 *Scientific American* column on tiling the plane periodically with congruent convex polygons (reprinted in my *Time Travel and Other Mathematical Bewilderments*) I promised a later column on nonperiodic tiling. This chapter reprints my fulfillment of that promise—a 1977 column that reported for the first time a remarkable nonperiodic tiling discovered by Roger Penrose, the noted British mathematical physicist and cosmologist. First, let me give some definitions and background.

A periodic tiling is one on which you can outline a region that tiles the plane by translation, that is, by shifting the position of the region without rotating or reflecting it. M. C. Escher, the Dutch artist, is famous for his many pictures of periodic tilings with shapes that resemble living things. Figure 1 is typical. An adjacent black and white bird constitute a fundamental region that tiles by translation. Think of the plane as being covered with transparent paper on which each tile is outlined. Only if the tiling is periodic can you shift the paper, without rotation, to a new position where all outlines again exactly fit.

An infinity of shapes—for instance the regular hexagon—tile only periodically. An infinity of other shapes tile both periodically and nonperiodically. A checkerboard is easily converted to a nonperiodic tiling by identical isosceles right triangles or by quadrilaterals. Simply bisect



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Figure 1 A periodic tessellation by M. C. Escher (1949)

each square as shown in Figure 2A, left, altering the orientations to prevent periodicity. It is also easy to tile nonperiodically with dominoes.

Isoceles triangles also tile in the radial fashion shown in the center of Figure 2A. Although the tiling is highly ordered, it is obviously not periodic. As Michael Goldberg pointed out in a 1955 paper titled “Central Tessellations,” such a tiling can be sliced in half, and then the half planes can be shifted one step or more to make a spiral form of nonperiodic tiling, as shown in Figure 2A, right. The triangle can be distorted in an infinity of ways by replacing its two equal sides with congruent lines, as shown at the left in Figure 2B. If the new sides have straight edges, the result is a polygon of 5, 7, 9, 11 . . . edges that tiles spirally. Figure 3 shows a striking pattern obtained in this way from a nine-sided polygon. It was first found by Heinz Voderberg in a complicated procedure. Goldberg’s method of obtaining it makes it almost trivial.

In all known cases of nonperiodic tiling by congruent figures the figure also tiles periodically. Figure 2B, right, shows how two of the Voderberg enneagons go together to make an octagon that tiles periodically in an obvious way.

Another kind of nonperiodic tiling is obtained by tiles that group together to form larger replicas of themselves. Solomon W. Golomb calls them “reptiles.” (See Chapter 19 of my book *Unexpected Hanging*.) Figure 4 shows how a shape called the “sphinx” tiles nonperiodically by

giving rise to ever larger sphinxes. Again, two sphinxes (with one sphinx rotated 180 degrees) tile periodically in an obvious way.

Are there sets of tiles that tile only nonperiodically? By “only” we mean that neither a single shape or subset nor the entire set tiles periodically, but that by using all of them a nonperiodic tiling is possible. Rotating and reflecting tiles are allowed.

For many decades experts believed no such set exists, but the supposition proved to be untrue. In 1961 Hao Wang became interested in tiling the plane with sets of unit squares whose edges were colored in various ways. They are called Wang dominoes, and Wang wrote a splendid article about them for *Scientific American* in 1965. Wang’s problem was to find a procedure for deciding whether any given set of dominoes will tile by placing them so that abutting edges are the same color. Rotations and reflections are not allowed. The problem is important because it relates to decision questions in symbolic logic. Wang conjectured that any set of tiles which can tile the plane can tile it periodically and showed that if this is the case, there is a decision procedure for such tiling.

In 1964 Robert Berger, in his thesis for a doctorate from Harvard University in applied mathematics, showed that Wang’s conjecture is

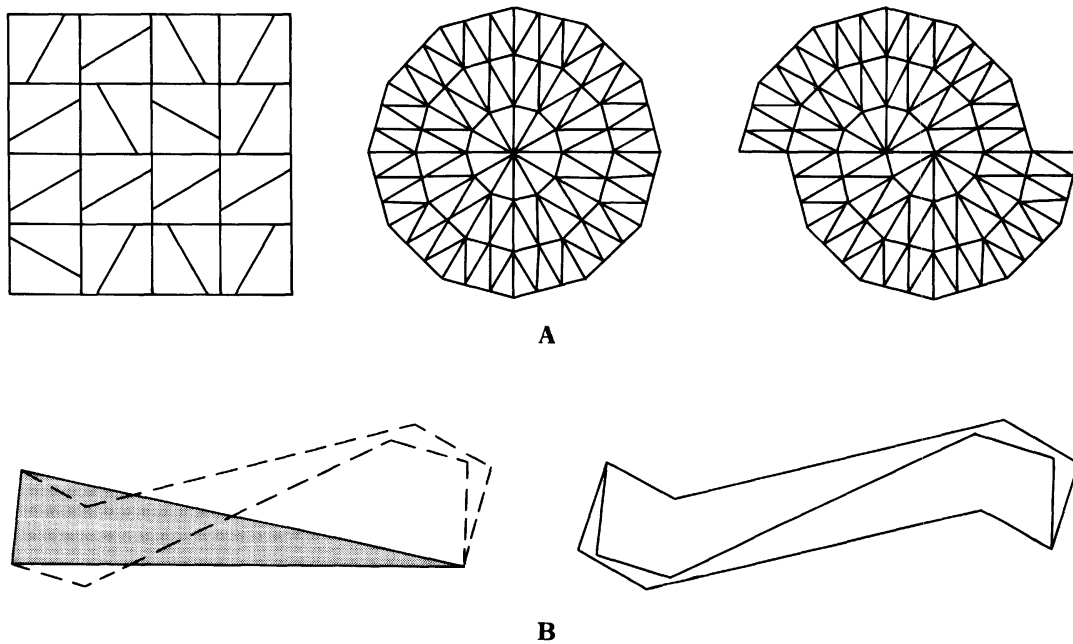


Figure 2 (A) Nonperiodic tiling with congruent shapes (B) An enneagon (dotted at left) and a pair of enneagons (right) forming an octagon that tiles periodically

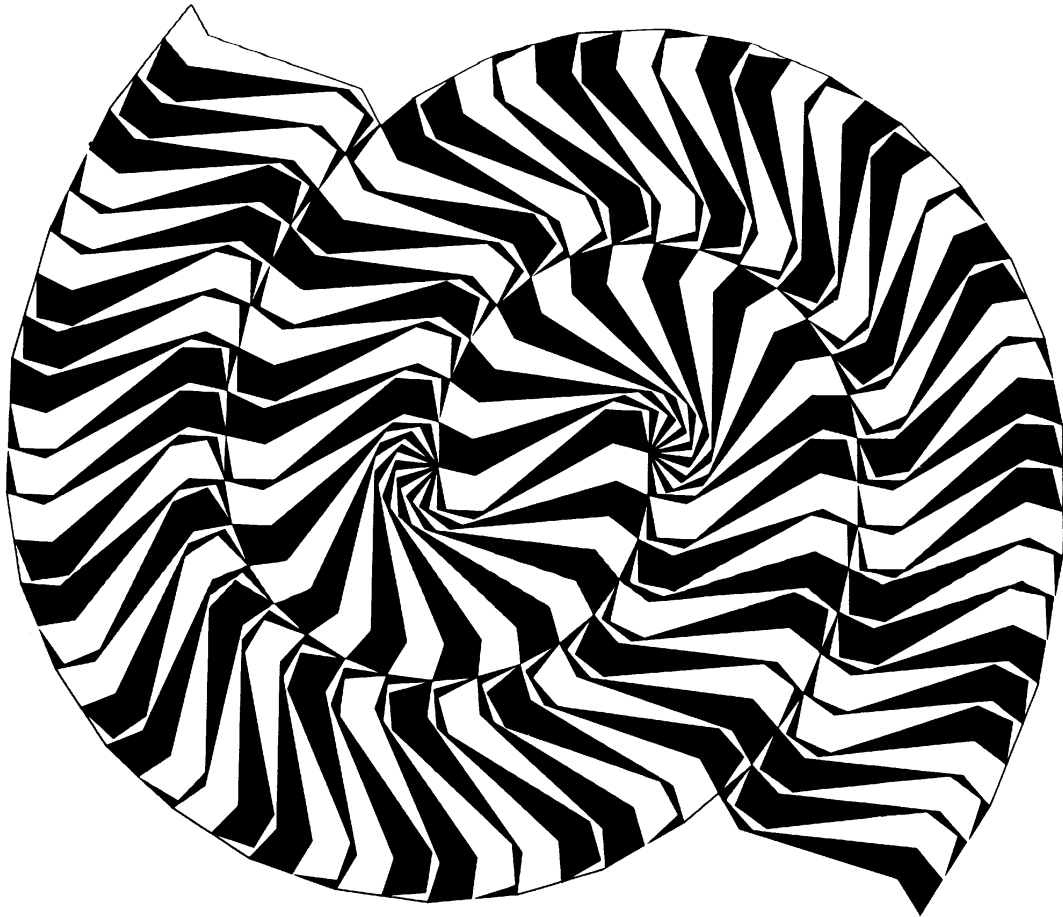


Figure 3 A spiral tiling by Heinz Voderberg

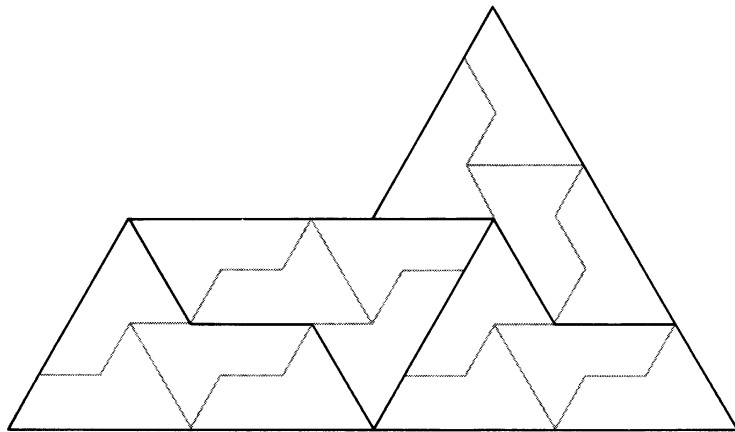


Figure 4 Three generations of sphinxes in a nonperiodic tiling

false. There is no general procedure. Therefore there is a set of Wang dominoes that tiles only nonperiodically. Berger constructed such a set, using more than 20,000 dominoes. Later he found a much smaller set of 104, and Donald Knuth was able to reduce the number to 92.

It is easy to change such a set of Wang dominoes into polygonal tiles that tile only nonperiodically. You simply put projections and slots on the edges to make jigsaw pieces that fit in the manner formerly prescribed by colors. An edge formerly one color fits only another formerly the same color, and a similar relation obtains for the other colors. By allowing such tiles to rotate and reflect Robinson constructed six tiles (see Figure 5) that force nonperiodicity in the sense explained above. In 1977 Robert Ammann found a different set of six tiles that also force nonperiodicity. Whether tiles of this square type can be reduced to less than six is not known, though there are strong grounds for believing six to be the minimum.

At the University of Oxford, where he is Rouse Ball Professor of Mathematics, Penrose found small sets of tiles, not of the square type,

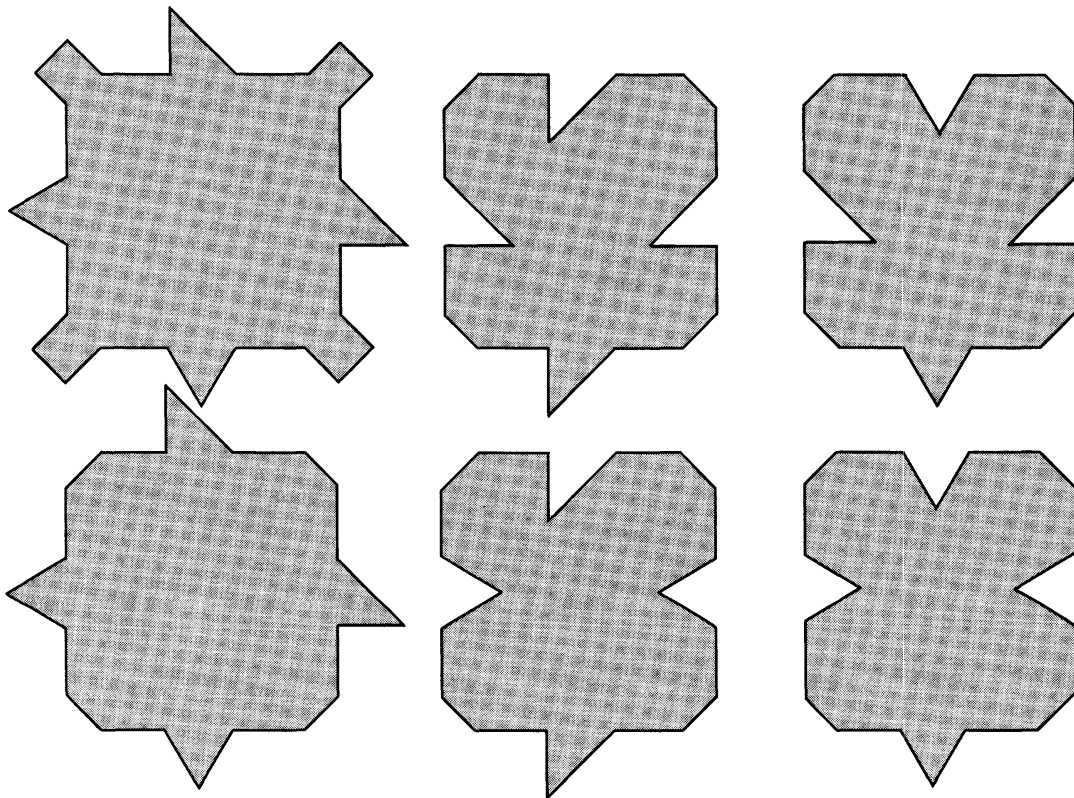


Figure 5 Raphael M. Robinson's six tiles that force a nonperiodic tiling

that force nonperiodicity. Although most of his work is in relativity theory and quantum mechanics, he continues the active interest in recreational mathematics he shared with his geneticist father, the late L. S. Penrose. (They are the inventors of the famous “Penrose staircase” that goes round and round without getting higher; Escher depicted it in his lithograph “Ascending and Descending.”) In 1973 Penrose found a set of six tiles that force nonperiodicity. In 1974 he found a way to reduce them to four. Soon afterward he lowered them to two.

Because the tiles lend themselves to commercial puzzles, Penrose was reluctant to disclose them until he had applied for patents in the United Kingdom, the United States and Japan. The patents are now in force. I am equally indebted to John Horton Conway for many of the results of his study of the Penrose tiles.

The shapes of a pair of Penrose tiles can vary, but the most interesting pair have shapes that Conway calls “darts” and “kites.” Figure 6A shows how they are derived from a rhombus with angles of 72 and 108 degrees. Divide the long diagonal in the familiar golden ratio of $(1 +$

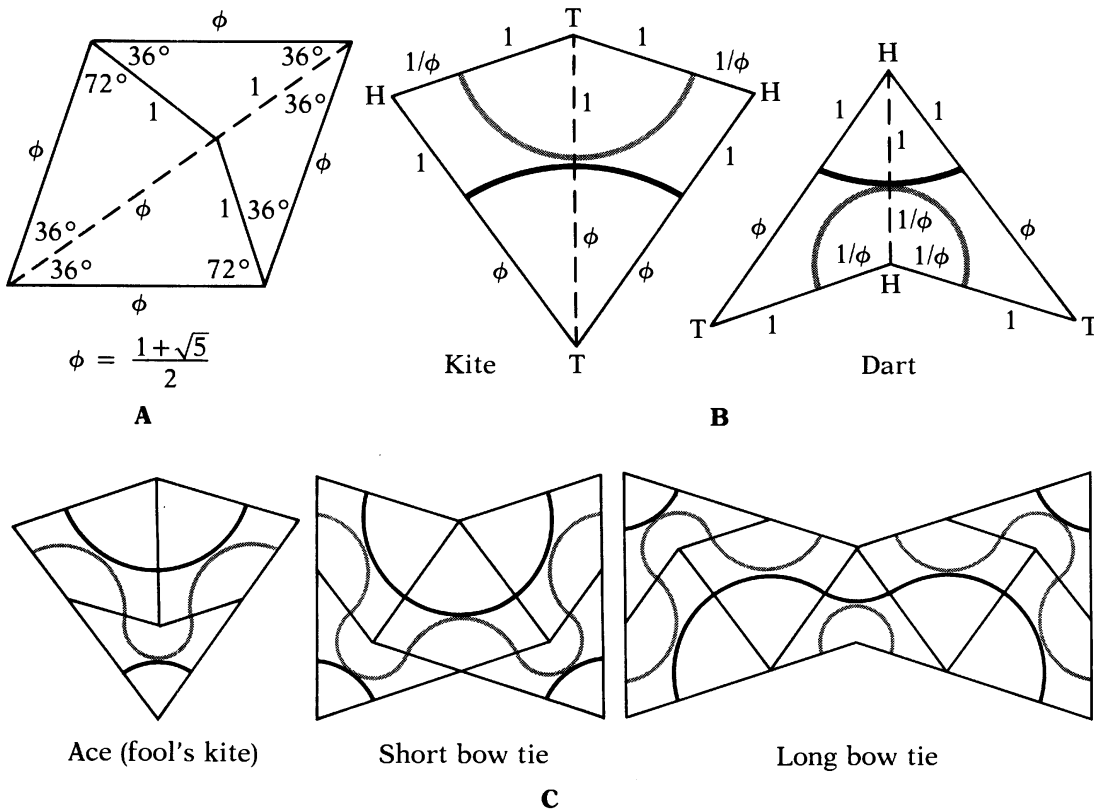


Figure 6 (A) Construction of dart and kite (B) A coloring (black and gray) of dart and kite to force nonperiodicity (C) Aces and bow ties that speed constructions

$\sqrt{5})/2 = 1.61803398 \dots$, then join the point to the obtuse corners. That is all. Let phi stand for the golden ratio. Each line segment is either 1 or phi as indicated. The smallest angle is 36 degrees, and the other angles are multiples of it.

The rhombus of course tiles periodically, but we are not allowed to join the pieces in this manner. Forbidden ways of joining sides of equal length can be enforced by bumps and dents, but there are simpler ways. For example, we can label the corners *H* and *T* (heads and tails) as is shown in Figure 6B, and then give the rule that in fitting edges only corners of the same letter may meet. Dots of two colors could be placed in the corners to aid in conforming to this rule, but a prettier method, proposed by Conway, is to draw circular arcs of two colors on each tile, shown in the illustration as black and gray. Each arc cuts the sides as well as the axis of symmetry in the golden ratio. Our rule is that abutting edges must join arcs of the same color.

To appreciate the full beauty and mystery of Penrose tiling one should make at least 100 kites and 60 darts. The pieces need be colored on one side only. The number of pieces of the two shapes are (like their areas) in the golden ratio. You might suppose you need more of the smaller darts, but it is the other way around. You need 1.618 . . . as many kites as darts. In an infinite tiling this proportion is exact. The irrationality of the ratio underlies a proof by Penrose that the tiling is nonperiodic because if it were periodic, the ratio clearly would have to be rational.

A good plan is to draw as many darts and kites as you can on one sheet, with a ratio of about five kites to three darts, using a thin line for the curves. The sheet can be photocopied many times. The curves can then be colored with, say, red and green felt-tip pens. Conway has found that it speeds constructions and keeps patterns stabler if you make many copies of the three larger shapes as is shown in Figure 6C. As you expand a pattern, you can continually replace darts and kites with aces and bow ties. Actually an infinity of arbitrarily large *pairs* of shapes, made up of darts and kites, will serve for tiling any infinite pattern.

A Penrose pattern is made by starting with darts and kites around one vertex and then expanding radially. Each time you add a piece to an edge, you must choose between a dart and a kite. Sometimes the choice is forced, sometimes it is not. Sometimes either piece fits, but later you may encounter a contradiction (a spot where no piece can be legally added) and be forced to go back and make the other choice. It is a good plan to go around a boundary, placing all the forced pieces first. They cannot lead to a contradiction. You can then experiment with unforced

pieces. It is always possible to continue forever. The more you play with the pieces, the more you will become aware of “forcing rules” that increase efficiency. For example, a dart forces two kites in its concavity, creating the ubiquitous ace.

There are many ways to prove that the number of Penrose tilings is uncountable, just as the number of points on a line is. These proofs rest on a surprising phenomenon discovered by Penrose. Conway calls it “inflation” and “deflation.” Figure 7 shows the beginning of inflation. Imagine that every dart is cut in half and then all short edges of the original pieces are glued together. The result: a new tiling (shown in heavy black lines) by larger darts and kites.

Inflation can be continued to infinity, with each new “generation” of pieces larger than the last. Note that the second-generation kite, although it is the same size and shape as a first-generation ace, is formed differently. For this reason the ace is also called a fool’s kite. It should never be mistaken for a second-generation kite. Deflation is the same process carried the other way. On every Penrose tiling we can draw smaller and smaller generations of darts and kites. This pattern too goes to infinity, creating a structure that is a fractal (see Chapter 3).

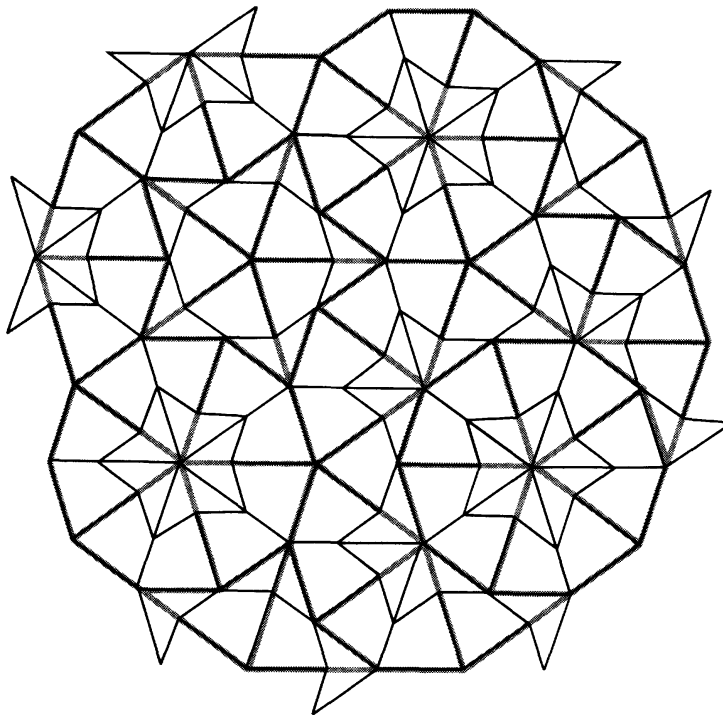


Figure 7 How a pattern is inflated

Conway's proof of the uncountability of Penrose patterns (Penrose had earlier proved it in a different way) can be outlined as follows. On the kite label one side of the axis of symmetry L , the other R (for left and right). Do the same on the dart, using l and r . Now pick a random point on the tiling. Record the letter that gives its location on the tile. Inflate the pattern one step, note the location of the same point in a second-generation tile and again record the letter. Continuing through higher inflations, you generate an infinite sequence of symbols that is a unique labeling of the original pattern seen, so to speak, from the selected point.

Pick another point on the original pattern. The procedure may give a sequence that starts differently, but it will reach a letter beyond which it agrees to infinity with the former sequence. If there is no such agreement beyond a certain point, the two sequences label distinct patterns. Not all possible sequences of the four symbols can be produced this way, but those that label different patterns can be shown to correspond in number with the number of points on a line.

We have omitted the colored curves on our pictures of tilings because they make it difficult to see the tiles. If you work with colored tiles, however, you will be struck by the beautiful designs created by these curves. Penrose and Conway independently proved that whenever a curve closes, it has a pentagonal symmetry, and the entire region within the curve has a fivefold symmetry. At the most a pattern can have two curves of each color that do not close. In most patterns all curves close.

Although it is possible to construct Penrose patterns with a high degree of symmetry (an infinity of patterns have bilateral symmetry), most patterns, like the universe, are a mystifying mixture of order and unexpected deviations from order. As the patterns expand, they seem to be always striving to repeat themselves but never quite managing it. G. K. Chesterton once suggested that an extraterrestrial being, observing how many features of a human body are duplicated on the left and the right, would reasonably deduce that we have a heart on each side. The world, he said, "looks just a little more mathematical and regular than it is; its exactitude is obvious, but its inexactitude is hidden; its wildness lies in wait." Everywhere there is a "silent swerving from accuracy by an inch that is the uncanny element in everything . . . a sort of secret treason in the universe." The passage is a nice description of Penrose's planar worlds.

There is something even more surprising about Penrose universes. In a curious finite sense, given by the "local isomorphism theorem," all Penrose patterns are alike. Penrose was able to show that every finite

region in any pattern is contained somewhere inside every other pattern. Moreover, it appears infinitely many times in every pattern.

To understand how crazy this situation is, imagine you are living on an infinite plane tessellated by one tiling of the uncountable infinity of Penrose tilings. You can examine your pattern, piece by piece, in ever expanding areas. No matter how much of it you explore you can never determine which tiling you are on. It is no help to travel far out and examine disconnected regions, because all the regions belong to one large finite region that is exactly duplicated infinitely many times on all patterns. Of course, this is trivially true of any periodic tessellation, but Penrose universes are not periodic. They differ from one another in infinitely many ways, and yet it is only at the unobtainable limit that one can be distinguished from another.

Suppose you have explored a circular region of diameter d . Call it the “town” where you live. Suddenly you are transported to a randomly chosen parallel Penrose world. How far are you from a circular region that exactly matches the streets of your home town? Conway answers with a truly remarkable theorem. The distance from the perimeter of the home town to the perimeter of the duplicate town is never more than d times half of the cube of the golden ratio, or $2.11+$ times d . (This is an upper bound, not an average.) If you walk in the right direction, you need not go more than that distance to find yourself inside an exact copy of your home town. The theorem also applies to the universe in which you live. Every large circular pattern (there is an infinity of different ones) can be reached by walking a distance in some direction that is certainly less than about twice the diameter of the pattern and more likely about the same distance as the diameter.

The theorem is quite unexpected. Consider an analogous isomorphism exhibited by a sequence of unpatterned digits such as pi. If you pick a finite sequence of 10 digits and then start from a random spot in pi, you are pretty sure to encounter the same sequence if you move far enough along pi, but the distance you must go has no known upper bound, and the expected distance is enormously longer than 10 digits. The longer the finite sequence is, the farther you can expect to walk to find it again. On a Penrose pattern you are always very close to a duplicate of home.

There are just seven ways that darts and kites will fit around a vertex. Let us consider first, using Conway’s nomenclature, the two ways with pentagonal symmetry.

The sun (shown in white in Figure 8) does not force the placing of any other piece around it. If you add pieces so that pentagonal symmetry

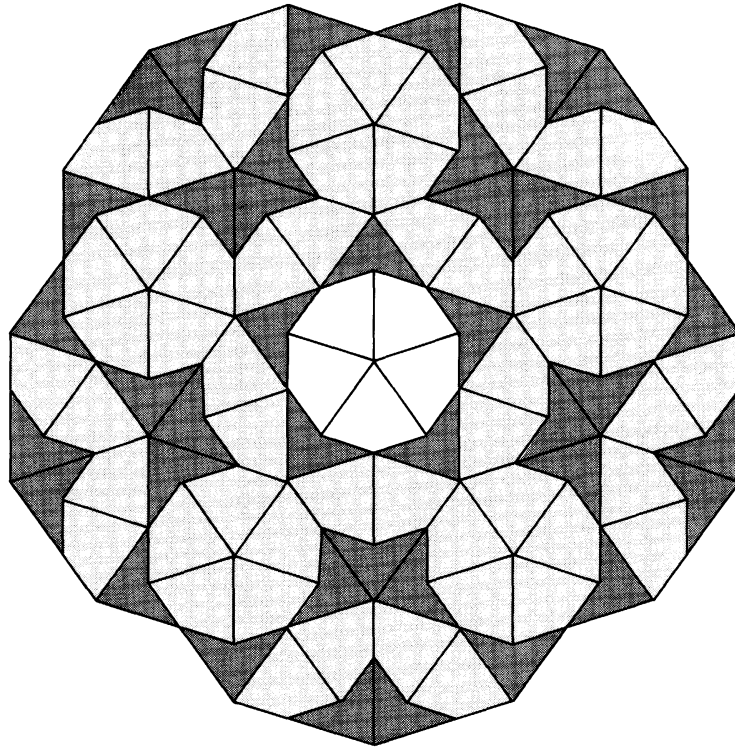


Figure 8 The infinite sun pattern

is always preserved, however, you will be forced to construct the beautiful pattern shown. It is uniquely determined to infinity.

The star, shown in white in Figure 9, forces the 10 light gray kites around it. Enlarge this pattern, always preserving the fivefold symmetry, and you will create another flowery design that is infinite and unique. The star and sun patterns are the only Penrose universes with perfect pentagonal symmetry, and there is a lovely sense in which they are equivalent. Inflate or deflate either of the patterns and you get the other.

The ace is a third way to tile around a vertex. It forces no more pieces. The deuce, the jack and the queen are shown in white in Figure 10, surrounded by the tiles they immediately force. As Penrose discovered (it was later found independently by Clive Bach), some of the seven vertex figures force the placing of tiles that are not joined to the immediately forced region. Plate 1 shows in deep color the central portion of the king's "empire." (The king is the dark gray area.) All the deep colored tiles are forced by the king. (Two aces, just outside the left and right borders, are also forced but are not shown.)

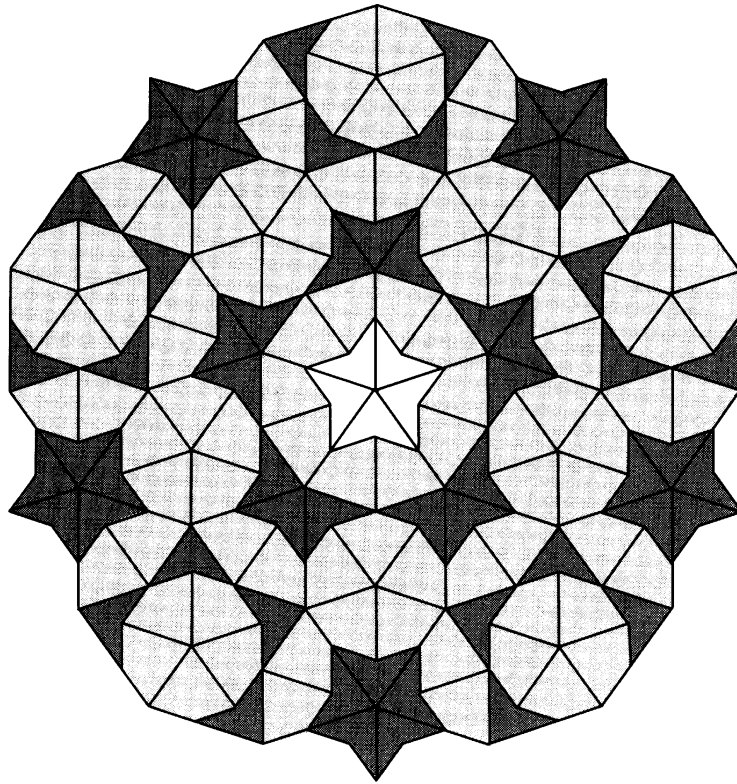


Figure 9 The infinite star pattern

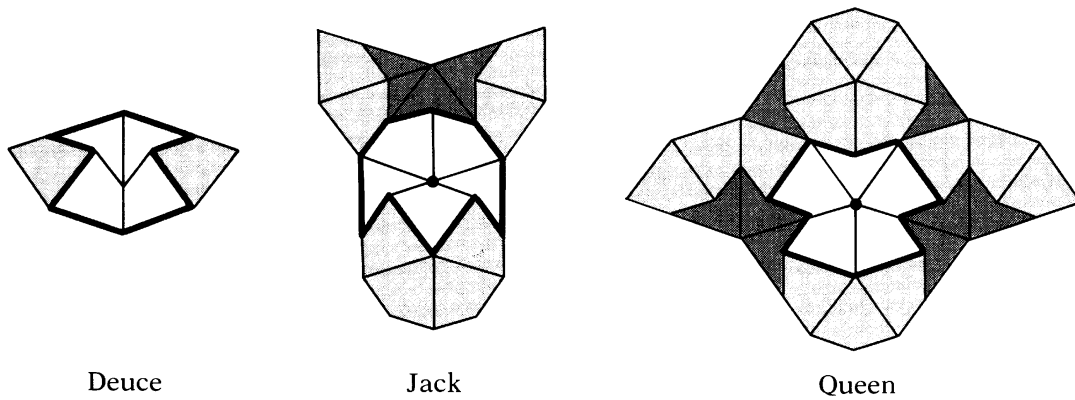


Figure 10 The "empires" of deuce, jack and queen