A CHARACTERIZATION OF FUCHSIAN ACTIONS BY TOPOLOGICAL RIGIDITY

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ABSTRACT. We prove that any rigid representation of $\pi_1\Sigma_g$ in $\mathrm{Homeo}^+(S^1)$ with Euler number at least g is necessarily semi-conjugate to a discrete, faithful representation into $\mathrm{PSL}(2,\mathbb{R})$. Combined with earlier work of Matsumoto, this precisely characterizes Fuchsian actions by a topological rigidity property. Though independent, this work can be read as an introduction to the companion paper [10] by the same authors.

1. INTRODUCTION

Let Σ_g be a surface of genus $g \geq 2$, and let $\Gamma_g = \pi_1(\Sigma_g)$. The *representation space* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ is the set of all actions of Γ_g on S^1 by orientation-preserving homeomorphisms, equipped with the compact-open topology. This is also the space of *flat topological circle bundles* over Σ_g, or equivalently, the space of circle bundles with a foliation transverse to the fibers. The *Euler class* of a representation $\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ is defined to be the Euler class of the associated bundle, and the *Euler number* $\mathrm{eu}(\rho)$ is the integer obtained by pairing the Euler class with the fundamental class of the surface. The classical Milnor-Wood inequality [13, 15] is the statement that the absolute value of the Euler number of a flat bundle is bounded by the absolute value of the Euler characteristic of the surface.

While the Euler number determines the topological type of a flat S^1 bundle, it doesn't nearly determine its flat structure – except in the special case where the Euler number is maximal, i.e. equal to $\pm(2g-2)$. In this case, a celebrated result of Matsuomoto states that for any representation ρ with $\mathrm{eu}(\rho) = \pm(2g-2)$, there there is a continuous, degree one, monotone map $h : S^1 \to S^1$ such that

$$h \circ \rho = \rho_F \circ h \tag{1}$$

where ρ_F is a discrete, faithful representation of Γ_g into $\mathrm{PSL}(2,\mathbb{R})$ (i.e. a bijection onto a cocompact lattice). We view $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Homeo}^+(S^1)$ via the action on $\mathbb{RP}^1 \cong S^1$ by Möbius transformations.

An important consequence of Matsumoto's theorem is that representations with maximal Euler number are dynamically *stable* or *rigid* in the following sense.

Definition 1.1. Let Γ be a discrete group. A representation $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ is called *path-rigid* if its path-component in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ consists of a single semi-conjugacy class.

Semi-conjugacy is the equivalence relation generated by the property shared by ρ and ρ_F in (1) above; we recall the precise definition in Section 2. As semi-conjugacy classes are connected in $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$, path-rigid representations are precisely those whose path-component is as small as possible. In fact, Matsumoto's work implies that maximal representations have the the stronger property (called "rigid" in [10]) of defining an isolated point in the *character space* for representations of Γ_g into $\mathrm{Homeo}^+(S^1)$; a notion of rigidity that generalizes to representations into arbitrary topological groups. However, for simplicity, we will not define character spaces here and refer the reader to [10] for details.

This paper proves the following converse to Matsumoto's rigidity result.

Theorem 1.2. *Let* $\rho : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ *be a path-rigid representation, with* $|\mathrm{eu}(\rho)| \geq g$. *Then* $\mathrm{eu}(\rho)$ *is maximal, i.e.* $|\mathrm{eu}(\rho)| = 2g-2$, *and* ρ *is semi-conjugate to a discrete, faithful representation into* $\mathrm{PSL}(2,\mathbb{R})$.

Thus, Fuchsian representations are characterized among all representations with Euler number at least g by path-rigidity. The assumption $|\mathrm{eu}(\rho)| \geq g$ is not superfluous – it is shown in [8] that many representations with $|\mathrm{eu}(\rho)| \leq g-1$ are path-rigid as well. However, our assumption can be replaced with an *a priori* strictly weaker assumption on the dynamics of ρ, phrased in terms of rotation numbers of elements, as follows.

Theorem 1.3. *Suppose* $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ *is path-rigid. If there exist based simple closed curves* $a, b \in \Gamma_g$ *with intersection number 1 and such that* $\widetilde{\mathrm{rot}}[\rho(a), \rho(b)] = \pm 1$, *then* $\mathrm{eu}(\rho) = \pm(2g-2)$, *and* ρ *is semi-conjugate to a Fuchsian representation.*

The hypothesis $\widetilde{\mathrm{rot}}[\rho(a), \rho(b)] = \pm 1$ is equivalent to the statement that the restriction of the representation to the torus defined by a and b is semi-conjugate to a geometric one (see [12]). Thus, one can think of the statement above as a local–to–global result: the local condition that a torus is Fuchsian, together with path-rigidity, implies the global statement that the representation is Fuchsian.

1.1. **Geometric representations.** If M is a manifold, and Γ a discrete group, a representation $\rho\colon \Gamma \to \mathrm{Homeo}^+(M)$ is called *geometric* if it is a bijection onto a cocompact lattice in a transitive, connected Lie group in $\mathrm{Homeo}(M)$.

It is not difficult to prove that the transitive Lie subgroups of $\mathrm{Homeo}^+(S^1)$ are precisely $\mathrm{SO}(2)$ and the central extensions of $\mathrm{PSL}(2,\mathbb{R})$ by finite cyclic groups (see [6]). Thus, the geometric representations $\Gamma_g \to \mathrm{Homeo}^+(S^1)$ are either Fuchsian or obtained by lifting a Fuchsian representation to one of these central extensions of $\mathrm{PSL}(2,\mathbb{R})$. The main result of [8] implies that all geometric representations are path-rigid (in fact, the proof shows the stronger result that they are rigid in the character space sense), and it was conjectured there the converse held as well.

This paper proves that conjecture under an additional assumption, which rules out the case that ρ could be a lattice in a nontrivial central extension of $\mathrm{PSL}(2,\mathbb{R})$. This assumption greatly simplifies the situation, allowing us to give a short proof of the converse to Matsumoto's result. The general converse is the subject of our recent paper [10]. Though self-contained and independent, the present work is also intended to serve as an introduction to the ideas in [10], communicating some of the philosophy of the proof in a simplified setting that avoids much of the technical nightmare. In this spirit, we have taken care to make the proof here as explicit and elementary as possible.

1.2. **Outline.** In Section 2 we recall standard material on dynamics of groups acting on the circle, including rotation numbers and the Euler number for actions of surface groups. We then introduce important tools for the proof of Theorem 1.3, and give a quick proof that Theorem 1.3 implies Theorem 1.2.

Sections 3 through 5 are devoted to the proof of Theorem 1.3. The general strategy is as follows. Given a representation ρ satisfying the hypotheses of Theorem 1.3, we show that:

1. After modifying ρ by a semi-conjugacy, there exists $a \in \Gamma_g$ represented by a nonseparating simple closed curve and such that $\rho(a)$ is *hyperbolic*, meaning that it is conjugate to a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$.
2. Using step 1, we then show that (again after semi-conjugacy of ρ), *any* $\gamma \in \Gamma_g$ represented by a nonseparating simple closed curve has the property that $\rho(\gamma)$ is hyperbolic. These two first steps are done in Section 3.
3. Next, we start to "reconstruct the surface", showing that the arrangement of attracting and repelling points of hyperbolic elements $\rho(\gamma)$, as γ ranges over simple closed curves, mimics that of a Fuchsian represenation into $\mathrm{PSL}(2,\mathbb{R})$. This is carried out in Section 4.
4. Finally, in Section 5 we show that the restriction of ρ to small subsurfaces is semi-conjugate to a Fuchsian representation; this is then improved to a global result by additivity of the *relative Euler class*.

Throughout this paper, whenever we say "deformation", we mean deformation along a continuous path in $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

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2. Preliminaries

This section gives a very quick review of basic concepts used later in the text. The only material that is not standard is the *based intersection number* discussed in Section 2.4.

2.1. Rotation numbers and the Euler number.

Let $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ denote the group of homeomorphisms of $\mathbb{R}$ that commute with integer translations, this is a central extension of $\mathrm{Homeo}^+(S^1)$ by $\mathbb{Z}$. The primary dynamical invariant of such homeomorphisms is the translation or rotation number, whose use can be traced back to work of Poincaré [14].

Definition 2.1 (Poincaré)**.** Let $\widetilde{g} \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ and $x \in \mathbb{R}$. The *translation number* $\widetilde{\mathrm{rot}}(\widetilde{g})$ is defined by $\widetilde{\mathrm{rot}}(\widetilde{g}) := \lim\limits_{n\to\infty} \frac{\widetilde{g}^n(x)}{n}$, where x is any point in $\mathbb{R}$. For $g \in \mathrm{Homeo}^+(S^1)$, the ($\mathbb{R}/\mathbb{Z}$-valued) *rotation number* of g is defined by $\mathrm{rot}(g) := \widetilde{\mathrm{rot}}(\widetilde{g}) \bmod \mathbb{Z}$, where $\widetilde{g}$ is any lift of g.

It is a standard exercise to show that these limits exist, and are independent of the choice of point x. Note that $\mathrm{rot}(g)$ is also independent of the choice of lift $\widetilde{g}$, and that rot is invariant under conjugacy. (In fact it is invariant under semi-conjugacy as well.)

One way of defining the Euler number of a representation is in terms of translation numbers. This was perhaps first observed by Milnor and Wood [13, 15], who showed the following. For the purposes of this work, the reader may take this as the *definition* of Euler number.

Proposition 2.2. *Consider a standard presentation*

$$\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_i [a_i, b_i] \rangle.$$

Let $\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$*, and let* $\widetilde{\rho(a_i)}$ *and* $\widetilde{\rho(b_i)}$ *be any lifts of* $\rho(a_i)$ *and* $\rho(b_i)$ *to* $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$*. Then the* Euler number $\mathrm{eu}(\rho)$ *is given by*

$$\mathrm{eu}(\rho) = \widetilde{\mathrm{rot}}\left([\widetilde{\rho(a_1)}, \widetilde{\rho(b_1)}] \cdots [\widetilde{\rho(a_g)}, \widetilde{\rho(b_g)}] \right).$$

Note that, for any f and $g \in \mathrm{Homeo}^+(S^1)$, the value of the commutator $[\widetilde{f}, \widetilde{g}]$ is independent of the choice of lifts $\widetilde{f}$ and $\widetilde{g}$ in $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$.

As remarked in the introduction, the Milnor–Wood inequality is the statement that $|\mathrm{eu}(\rho)| \leq 2g - 2$. For a simpler surface as a one-holed torus, we have the following variation, which was essentially proved in [15]; see the discussion following Lemma 5.2 below.

Lemma 2.3. *Let* $[\widetilde{f}, \widetilde{g}] \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ *be a commutator. Then* $-1 \leq \widetilde{\mathrm{rot}}([\widetilde{f}, \widetilde{g}]) \leq 1$.

Though unimportant in the preceding remarks, in what follows we will need to fix a convention for commutators and group multiplication.

Convention 2.4. *We read words in* Γ_g *from right to left, so that group multiplication coincides with function composition. (This is convenient for dealing with representations to* $\mathrm{Homeo}^+(S^1)$*.) We set the notation for a commutator as*

$$[a, b] := b^{-1} a^{-1} b a.$$

2.2. **Dynamics of groups acting on S^1.** The material in this section is covered in more detail in [6] and [9].

Definition 2.5 (Ghys [5])**.** Let Γ be a group. Two representations ρ_1, ρ_2 in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ are *semi-conjugate* if there is a monotone (possibly non-continuous or non-injective) map $\widetilde{h} \colon \mathbb{R} \to \mathbb{R}$ such that $\widetilde{h}(x+1) = \widetilde{h}(x) + 1$ for all $x \in \mathbb{R}$, and such that, for all $\gamma \in \Gamma$, there are lifts $\widetilde{\rho_1(\gamma)}$ and $\widetilde{\rho_2(\gamma)}$ such that $\widetilde{h} \circ \widetilde{\rho_1(\gamma)} = \widetilde{\rho_2(\gamma)} \circ \widetilde{h}$.

Ghys [5] showed that semi-conjugacy is an equivalence relation on $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ (see also [9] for an exposition of the proof). In fact, it is the equivalence relation generated by the relationship shared by ρ and ρ_F in Equation (1) of Section 1.

The next proposition states a useful dynamical trichotomy for groups acting on the circle, which in particular can be used to explain when a semi-conjugacy map can be taken to be continuous. As it is classical, we do not repeat the proof; the reader may refer to [6, Prop. 5.6].

Proposition 2.6. *Let* $G \subset \mathrm{Homeo}^+(S^1)$*. Then exactly one of the following holds:*

i) *G has a finite orbit.*
ii) *G is* minimal*, meaning that all orbits are dense.*
iii) *There is a unique compact G-invariant subset of S^1 contained in the closure of any orbit, on which G acts minimally. This set is homeomorphic to a Cantor set and called the* exceptional minimal set *for G.*

In case *iii*), defining h to be a map that collapses each interval in the complement of the exceptional minimal set to a point gives the following (we leave the proof as an exercise).

Proposition 2.7. *Let $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ be a homomorphism such that $\rho(G)$ has an exceptional minimal set. Then ρ is semi-conjugate to a homomorphism ν whose image is minimal. Moreover, provided that ν is minimal, any semi-conjugacy h to any representation ρ' such that $h \circ \rho' = \nu \circ h$ is necessarily continuous.*

We will make frequent use of the following two consequences of Proposition 2.7.

Corollary 2.8. *Suppose that ρ and ρ' are semi-conjugate representations. If both ρ and ρ' are minimal, then they are* conjugate.

Corollary 2.9. *Let $\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ be a path-rigid representation. Then ρ is semi-conjugate to a minimal representation.*

Proof. Corollary 2.8 follows immediately from Proposition 2.7. We now prove Corollary 2.9. Using Propositions 2.6 and 2.7, it suffices to show that a representation with a finite orbit is not path-rigid. If ρ has a finite orbit, then we may perform the Alexander trick to continuously deform ρ into a representation with image in $\mathrm{SO}(2)$. As $\mathrm{Hom}(\Gamma_g, \mathrm{SO}(2)) = \mathrm{SO}(2)^{2g}$, the representation ρ can be deformed arbitrarily within this space, in particular to a non semi-conjugate representation. □

Following Corollary 2.9, in the proof of Theorem 1.3 we will occasionally make the (justified) assumption that a path-rigid representation ρ is also minimal.

2.3. **Deforming actions of surface groups.** Let $\gamma \in \Gamma_g$ be a based, simple loop. Cutting Σ_g along γ decomposes Γ_g into an amalgamated product $\Gamma_g = A *_{\langle\gamma\rangle} B$ if γ is separating, and an HNN-extension $A*_{\langle\gamma\rangle}$ if not. In both cases, A and B are free groups. As there is no obstruction to deforming a representations of a free group into any topological group, deforming a representation $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ amounts to deforming the restriction(s) of ρ on A (and B, if γ separates), subject to the single constraint that these should agree on γ.

The following explicit deformations are analogous to special cases of *bending deformations* from the theory of quasi-Fuchsian and Kleinian groups.

Definition 2.10. (Bending deformations)

(1) *Separating curves.* Let $\gamma = c \in \Gamma_g$ represent a separating simple closed curve with $\Gamma_g = A *_{\langle c\rangle} B$. Let c_t be a one-parameter group of homeomorphisms commuting with $\rho(c)$. Define ρ_t to agree with ρ on A, and to be equal to $c_t \rho c_t^{-1}$ on B.
(2) *Nonseparating curves.* Let $\gamma = a$, and let $b \in \Gamma_g$ with $i(a,b) = -1$. Let $c = [a,b]$, writing again $\Gamma_g = A *_{\langle c\rangle} B$. Let a_t be 1-parameter group commuting with $\rho(a)$ and define ρ_t to agree with ρ on B, and on $\langle a\rangle$, and define $\rho_t(b) = a_t\rho(b)$.

In both cases, we call this deformation a *bending along* γ.

In particular, if γ_t is a one-parameter group with $\gamma_1 = \gamma$, then the deformation given above is the precomposition of ρ with τ_{γ_*}, where τ_γ is the Dehn twist along γ. Note that we have made a specific (though arbitrary) choice realizing the Dehn twist as an automorphism of Γ_g. This will allow us to do specific computations, for which having a twist defined only up to inner automorphism would not suffice. (See the discussion on based curves in the next subsection for more along these lines.) While not every $f \in \mathrm{Homeo}^+(S^1)$ embeds in a one-parameter group, every element with at least one fixed point does, and this is the situation in which we will typically apply bending deformations in this article.

The next corollary is used frequently in the proof of Theorem 1.3.

Corollary 2.11. *Suppose that ρ is a path-rigid, minimal representation. Let ρ_t be a bending deformation along a, using a deformation a_t, with $a_1 = a^N$ for some $N \in \mathbb{Z}$. Then ρ_1 is conjugate to ρ.*

Proof. By the discussion above, ρ_1 agrees with precomposition of ρ with an automorphism of Γ_g, so has the same image. Corollary 2.8 now implies that these are conjugate. $\square$

2.4. **Based curves, chains, and Fuchsian tori.** If a and b are simple closed curves on Σ_g, the familiar *geometric intersection number* is the minimum value of $|a' \cap b'|$, where a' and b' are any curves freely homotopic to a and b respectively. It is well known that if a and b are nonseparating simple closed curves with geometric intersection number 1, then there is a subsurface $T \subset \Sigma$ homeomorphic to a torus with one boundary component with fundamental group (freely) generated by a and b. (See e.g. [4] Section 1.2.3.)

As mentioned earlier, the fact that we are working with specific representations, rather than conjugacy classes of elements, forces us to take basepoint and orientation of curves into account. Although our notation $\Gamma_g = \pi_1\Sigma_g$ does not mention a basepoint, all elements of $\pi_1\Sigma_g$ will henceforth always be assumed based, and we will use the following variation on the standard definition of intersection number.

Definition 2.12 (Based intersection number)**.** Let $a, b \in \Gamma_g$. We write $i(a, b) = 0$ if we can represent a and b by differentiable maps $a, b\colon [0, 1] \to \Sigma_g$, based at the base point, whose restrictions to $[0, 1)$ are injective, and such that the cyclic order of their tangent vectors at the base point is either $(a'(0), -a'(1), b'(0), -b'(1))$ or $(a'(0), -a'(1), -b'(1), b'(0))$, or the reverse of one of these.
If instead the cyclic order of tangent vectors is $(a'(0), b'(0), -a'(1), -b'(1))$ or the reverse, we write $i(a, b) = 1$ and $i(a, b) = -1$ respectively.

This is a somewhat ad-hoc definition. In particular, $i(a, b)$ is left undefined for many pairs (a, b).

For more than two curves, the following definition will be convenient.

Definition 2.13. A *directed k-chain*, in Σ_g, is a k-uple $(\gamma_1, \dots, \gamma_k)$ of elements of Γ_g, such that the oriented curves γ_i may be homotoped simultaneously (rel. the base point) in order to realize an embedding (possibly orientation-reversing, but respecting the orientations of the edges) of the graph shown in Figure 1.

In particular, $i(\gamma_i, \gamma_j) = \pm 1$ if $|j - i| = 1$, and 0 otherwise. Note that we do not

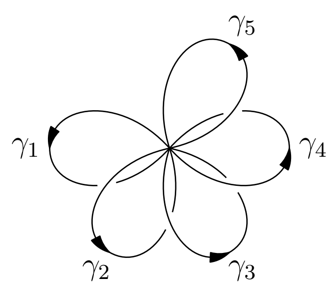

FIGURE 1. A directed chain of length 5

require that the embedding be π_1-injective. For example, whenever $i(\gamma_1, \gamma_2) = 1$, then $(\gamma_1, \gamma_2, \gamma_1^{-1})$ is a directed 3-chain, rather degenerate.

These k-chains will be useful especially to study bending deformations that realize sequences of Dehn twists. Whenever $(\gamma_1, \dots, \gamma_k)$ is a directed k-chain, the Dehn twist along the curve γ_i may be described by an automorphism of Γ_g leaving invariant the elements γ_j for $|j-i| \geq 2$ and $j = i$, and mapping γ_{i-1} to $\gamma_i^{-1}\gamma_{i-1}$, and γ_{i+1} to $\gamma_{i+1}\gamma_i$.

Notation 2.14. Let $i(a,b) = \pm 1$. Then their commutator $[a,b]$ bounds a genus 1 subsurface (well defined up to homotopy) containing a and b. We denote this surface by $T(a,b)$.

Definition 2.15. We call any representation $\rho\colon \pi_1(T(a,b)) \to \mathrm{PSL}(2,\mathbb{R})$ arising from a hyperbolic structure of infinite volume on $T(a,b)$ a *standard Fuchsian representation of a once-punctured torus.* Similarly, we say that $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ is *standard Fuchsian* if it comes from a hyperbolic structure on Σ_g.

Convention 2.16. *We assume Σ_g is oriented, hence standard Fuchsian representations of Γ_g have Euler number $-2g+2$, and are all conjugate in* $\mathrm{Homeo}^+(S^1)$. *Similarly, $T(a,b)$ inherits and orientation, so all its standard Fuchsian representations are conjugate in* $\mathrm{Homeo}^+(S^1)$.

Definition 2.17. We say that $\rho : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ has a *Fuchsian torus* if there exist two simple closed curves $a, b \in \Gamma_g$, with $i(a,b) = \pm 1$ and such that $\widetilde{\mathrm{rot}}([\rho(a), \rho(b)]) = \pm 1$.

The terminology "Fuchsian torus" in Definition 2.17 comes the following observation of Matsumoto.

Observation 2.18 ([11])**.** *Let $\alpha, \beta \in$ $\mathrm{Homeo}^+(S^1)$ be such that $\widetilde{\mathrm{rot}}([\alpha,\beta]) = \pm 1$. Then α and β generate a free group, and, up to reversing the orientation of S^1, this group is semi-conjugate to a standard Fuchsian representation of a one-holed torus $T(a,b)$ with $\rho(a) = \alpha$ and $\rho(b) = \beta$.*

The proof of Observation 2.18 not difficult, an easily readable sketch is given in [12, § 3].

The next lemma shows the existence of such a torus is guaranteed, provided the absolute value of the Euler number of a representation is sufficiently high.

Lemma 2.19. *If $|\mathrm{eu}(\rho)| \geq g$ then ρ has a Fuchsian torus.*

Proof. If $\mathrm{eu}(\rho) \geq g$, then conjugating ρ by an orientation-reversing homeomorphism of S^1 gives a representation with Euler number at most $-g$. Thus, we may assume that $\mathrm{eu}(\rho) \leq -g$. Let $f \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. It is an easy consequence of the definition of $\widetilde{\mathrm{rot}}$ that $\widetilde{\mathrm{rot}}(f) > 0$ if and only if $f(x) > x$ for all $x \in \mathbb{R}$. Hence if $f_1, \dots, f_g \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ satisfy $\widetilde{\mathrm{rot}}(f_i) > 0$ for all i, then $\widetilde{\mathrm{rot}}(f_1 \cdots f_g) > 0$.

By composing such f_i by translation by -1, which is central in $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, we deduce that if $\widetilde{\mathrm{rot}}(f_i) > -1$ for all i then $\widetilde{\mathrm{rot}}(f_1 \cdots f_g) > -g$. Now let ρ be a representation, and let $f_i = [\widetilde{\rho(a_i)}, \widetilde{\rho(b_i)}]$. Then the inequality $\mathrm{eu}(\rho) \leq -g$ implies $\widetilde{\mathrm{rot}}(f_i) \leq -1$ for some i. As the maximum absolute value of the rotation number of a commutator is 1 by Lemma 2.3, we in fact have $\widetilde{\mathrm{rot}}(f_i) = -1$ for some i. $\square$

Lemma 2.19 immediately shows that Theorem 1.3 implies Theorem 1.2. The rest of this work is devoted to the proof of Theorem 1.3.

3. Step 1: Existence of hyperbolic elements

Definition 3.1. We say a homeomorphism $f \in \mathrm{Homeo}^+(S^1)$ is *hyperbolic* if it is conjugate to a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$, i.e. it has one *attracting fixed point* $f_+ \in S^1$ and one *repelling fixed point* $f_- \neq f_+$ such that $\lim\limits_{n\to+\infty} f^n(x) = f_+$ for all $x \neq f_-$, and $\lim\limits_{n\to+\infty} f^{-n}(x) = f_-$ for all $x \neq f_+$.

The first step of the proof of Theorem 1.3 is to show that a rigid, minimal representation has very many hyperbolic elements. This is the goal of this section.

Lemma 3.2. *Let $T(a,b)$ be a one-holed torus subsurface, and let $A = \pi_1 T(a,b)$. Suppose $\rho\colon A \to \mathrm{Homeo}^+(S^1)$ is semi-conjugate to a standard Fuchsian representation, as in definition 2.15. Then there exists a continuous deformation ρ_t with $\rho_0 = \rho$ such that*

i) $\rho_1(a)$ is hyperbolic, and

ii) there exists a continuous family of homeomorphisms $f_t \in \mathrm{Homeo}^+(S^1)$ such that $\rho_t([a,b]) = f_t \rho([a,b]) f_t^{-1}$ for all t.

Proof. Let c denote the commutator $[a,b]$. Let ρ_0 denote the minimal representation (unique up to conjugacy) that is semi-conjugate to ρ. Since ρ is semi-conjugate to a standard Fuchsian representation, ρ_0 is the representation corresponding to the *finite volume* hyperbolic structure on $T(a,b)$. By Observation 2.18 and Proposition 2.7, there is a continuous map $h : S^1 \to S^1$, collapsing each component of the exceptional minimal set for ρ to a point, satisfying $h\rho = \rho_0 h$. Let x_+ and x_- be the endpoints of the axis of $\rho_0(a)$, and X_+ and X_- the pre-images under h of their orbits $\rho(A)x_+$ and $\rho(A)x_-$.

Note that X_+ and X_- are both $\rho(A)$-invariant sets and their images under h are the attractors (respectively, repellers) of closed curves in $T(a,b)$ conjugate to a. Moreover, for this reason, X_+ and X_- lie in a single connected component of $S^1 \smallsetminus \mathrm{Fix}(\rho(c))$, and the interiors of the intervals that make up X_+ and X_- are disjoint from the exceptional minimal set of ρ.

Define a continuous family of continuous maps $h_t : S^1 \to S^1$, with $h_0 = \mathrm{id}$, as follows: We define h_t to be the identity on the complement of the connected component of $S^1 \smallsetminus \mathrm{Fix}(\rho(c))$ containing X_+ and X_-, and for each interval I of X_+ or of X_-, have h_t be a homotopy contracting that interval so that $h_1(I)$ is a point.

To make this precise, one needs to fix an identification of the target of h_t with the standard unit circle. Let J be the connected component of $S^1 \smallsetminus \mathrm{Fix}(\rho(c))$ that contains the exceptional minimal set of $\rho(A)$. Define h_t to rescale the length of each connected component of X_+ or X_- by a factor of $(1-t)$ and rescale the complement of $X_+ \cup X_-$ in J so that the total length of J remains unchanged. This gives us the desired map h_t which is the identity outside of J, and contracts intervals of X_+ and X_- to points.

Now define ρ_t by $h_t \rho(g) h_t^{-1} = \rho_t(g)$ for $t \in [0,1)$. We claim that there is a unique $\rho_1(g)$ satisfying $h_1 \rho(g) = \rho_1(g) h_1$. Indeed, $\rho(g)$ permutes the complementary intervals of the exceptional minimal set for ρ, so letting $h_1^{-1}(x)$ denote the pre-image of x by h_1 (which is either a point or an open interval complementary to the exceptional minimal set), $h_1 \rho(g) h_1^{-1}(x)$ is always a single point, and $h_1 \rho(g) h_1^{-1}$ defines in this way a homeomorphism, which we denote by $\rho_1(g)$. It is easily verified that $\rho_t(g)$ approaches $\rho_1(g)$ as $t \to 1$. By construction, $\rho_1(a)$ is hyperbolic, and $\rho_t(c)$ is conjugate to a translation on the interval J defined above (and hence its restriction to J is conjugate to $\rho(c)|_J$), and $\rho_t(c)$ restricted to $S^1 \smallsetminus J$ agrees with $\rho_0(c)$. Let $f_t : S^1 \to S^1$ be a continuous family of homeomorphisms supported on

J that conjugate the action of $\rho_t([a,b])$ to the action of $\rho(c)$ there. (For the benefit of the reader, justification of this step via a simple construction of such a family is given in Lemma 3.3 below.) Then $\rho_t(c) = f_t \rho(c) f_t^{-1}$, as claimed. $\square$

Lemma 3.3. *Let g_t be a continuous family (though not necessarily a subgroup) of homeomorphisms of an open interval I, with* $\mathrm{Fix}(g_t) \cap I = \emptyset$ *for all $t \in [0,1]$. Then there exists a continuous family of homeomorphisms f_t such that $f_t g_t f_t^{-1} = g_0$ for all t.*

Proof. Fix x in the interior of I, and let $D_t := [x, g_t(x)]$ be a fundamental domain for the action of g_t. Define the restriction of f_t to D_0 be the (unique) affine homeomorphism $D_0 \to D_t$, and extend f_t equivariantly to give a homeomorphism of I. $\square$

Corollary 3.4. *Let $\rho : \Gamma_g \to \mathrm{Homeo}^+(S^1)$. Suppose that a and b are simple closed curves in Γ_g with $i(a,b) = \pm 1$ and $\widetilde{\mathrm{rot}}([\widetilde{\rho(a)}, \widetilde{\rho(b)}]) = \pm 1$. Then there exists a deformation ρ' of ρ such that $\rho'(a)$ is hyperbolic. If additionally ρ is assumed path-rigid and minimal, then $\rho(a)$ is hyperbolic.*

Proof. Let A denote the subgroup generated by a and b and let $c = [a,b]$, so $\Gamma_g = A *_{\langle c \rangle} B$. Let $\bar{\rho}$ denote the restriction of ρ to A. By Lemma 3.2, there exists a family of representations $\bar{\rho}_t : A \to \mathrm{Homeo}^+(S^1)$ such that $\bar{\rho}_t(c) = f_t \bar{\rho}(c) f_t^{-1}$ for some continuous family $f_t \in \mathrm{Homeo}^+(S^1)$, and such that $\bar{\rho}_1(a)$ is hyperbolic. As in the bending construction, define a deformation of ρ by

$$\rho_t(\gamma) = \begin{cases} \bar{\rho}_t(\gamma) & \text{for } \gamma \in A \\ f_t \rho(\gamma) f_t^{-1} & \text{for } \gamma \in B. \end{cases}$$

By construction, ρ_t is a well defined representation, and $\rho_1(a) = \bar{\rho}_1(a)$ is hyperbolic.

If ρ is assumed path-rigid, then this deformation ρ' is semi-conjugate to ρ. If ρ is additionally known to be minimal, then there is a continuous map h satisfying $h \circ \rho' = \rho \circ h$. In particular, this implies that $\mathrm{Fix}(\rho(a)) = h\mathrm{Fix}(\rho'(a))$, so $\rho(a)$ has at most two fixed points. In this case, if $\rho(a)$ does not have hyperbolic dynamics then it has a lift to $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ satisfying $|x - \widetilde{\rho(a)}(x)| \leq 1$ for all x. But this easily implies that $|\widetilde{\mathrm{rot}}([\widetilde{\rho(a)}, \widetilde{\rho(b)}])| < 1$. (The reader may verify this as an exercise, or see the proof of Theorem 2.2 in [11] where this computation is carried out.) We conclude that $\rho(a)$ must be hyperbolic when ρ is path-rigid and minimal. $\square$

Having found one hyperbolic element, our next goal is to produce many others. An important tool here, and in what follows, is the following basic observation on dynamics of circle homeomorphisms.

Observation 3.5. *Let $f \in \mathrm{Homeo}^+(S^1)$ be hyperbolic, with attracting point f_+ and repelling point f_-, and let $g \in \mathrm{Homeo}^+(S^1)$. For any neighborhoods U_- and U_+ of f_- and f_+ respectively, and any neighborhoods V_- and V_+ of $g^{-1}(f_-)$ and $g(f_+)$ respectively, there exists $N \in \mathbb{N}$ such that*

$$f^N g(S^1 \smallsetminus V_-) \subset U_+ \text{ and } g f^N(S^1 \smallsetminus U_-) \subset V_+.$$

The proof is a direct consequence of Definition 3.1. Note that, if f is hyperbolic, then f^{-1} is as well (with attracting point f_- and repelling point f_+), so an analogous statement holds with f^{-1} in place of f and the roles of f_+ and f_- reversed.

We now state two useful consequences of this observation. The proofs are elementary and left to the reader.

Corollary 3.6. *Let $f \in \mathrm{Homeo}^+(S^1)$ be hyperbolic, and suppose g does not exchange the fixed points of f. Then for N sufficiently large, either $f^N g$ or $f^{-N} g$ has a fixed point.*

Corollary 3.7. *Let $f \in \mathrm{Homeo}^+(S^1)$ be hyperbolic, and suppose that $g^{-1}(f_-) \neq f_+$. Suppose also that $f^N g$ is known to be hyperbolic for large N. Then as $N \to \infty$, the attracting point of $f^N g$ approaches f_+ and the repelling point approaches $g^{-1}(f_-)$.*

With these tools in hand, we can use one hyperbolic element to find others.

Proposition 3.8. *Let ρ be path-rigid and minimal, and suppose that $i(a, b) = \pm 1$ and that $\rho(a)$ is hyperbolic. Then $\rho(b)$ is hyperbolic.*

Proof. We prove this under the assumption that $\rho(b)$ does not exchange the fixed points of $\rho(a)$. This assumption is justified by the next lemma (Lemma 3.9). Assuming $\rho(b)$ does not exchange the points of $\mathrm{Fix}(\rho(a))$, by Corollary 3.6, there exists some $N \in \mathbb{Z}$ such that $\rho(ba^N)$ has a fixed point. Since a is hyperbolic, a^N belongs to a 1-parameter family of homeomorphisms, and a bending deformation using this family gives a deformation ρ_1 of ρ with $\rho_1(b) = \rho(ba^N)$. By Corollary 2.11, using the fact that ρ is minimal, ρ_1 and ρ are conjugate. Thus, $\rho(b)$ has a fixed point and belongs to a 1-parameter group b_t.

Now we can build a bending deformation ρ'_t such that $\rho'_1(b) = \rho(b)$ and $\rho'_1(a) = \rho(ba)$. Thus, $\rho'_1(a^{-1}b) = \rho(a^{-1})$, which is hyperbolic. Since ρ'_1 and ρ are conjugate, this means that $\rho(a^{-1}b)$ is hyperbolic. Similarly, using the fact that a belongs to a one-parameter group, there exists a bending deformation ρ''_t with $\rho''_1(a^{-1}b) = \rho(b)$, and such that ρ''_1 is conjugate to ρ. This implies that $\rho(b)$ is hyperbolic. $\square$

Lemma 3.9. *Let $a, b \in \Gamma_g$ satisfy $i(a, b) = \pm 1$, and let $\rho \colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$. Suppose that $\rho(a)$ is hyperbolic, and $\rho(b)$ exchanges the fixed points of $\rho(a)$. Then there is a deformation ρ' of ρ which is not semi-conjugate to ρ.*

Proof. Note first that the property that $\rho(b)$ exchanges the fixed points of $\rho(a)$ implies that $\rho(b^{-1}a^{-1}b)$ is hyperbolic with the same attracting and repelling points as a. Hence $[\rho(a), \rho(b)]$ is hyperbolic with the same attracting and repelling points as well. We now produce a deformation ρ_1 of ρ such that $\rho_1(a)$ and $\rho_1(b)$ are in $\mathrm{PSL}(2, \mathbb{R})$, after this we will easily be able to make an explicit further deformation to a non semi-conjugate representation.

First, conjugate ρ so that $\rho(a) \in \mathrm{PSL}(2, \mathbb{R})$ and so that the attracting and repelling fixed points of $\rho(a)$ are at 0 and 1/2 respectively (thinking of S^1 as $\mathbb{R}/\mathbb{Z}$). Now choose a continuous path b_t from $b_0 = b$ to the order two rotation $b_1 : x \mapsto x + 1/2$, and such that $b_t(0) = 1/2$ and $b_t(1/2) = 0$ for all t. By the observation above, $[\rho(a), b_t]$ is hyperbolic with attracting fixed point 0 and repelling fixed point 1/2 for all t, so is conjugate to $\rho(a)$. By Lemma 3.3, applied separately to $(0, 1/2)$ and $(1/2, 1)$, there exists a continuous choice of conjugacies f_t such that $f_t[\rho(a), \rho(b)]f_t^{-1} = [\rho(a), b_t]$. Now to define ρ_t, we consider $\Gamma_g = A *_c B$ where $A = \langle a, b \rangle$ and $c = [a, b]$, and set

$$\begin{aligned} \rho_t(\gamma) &= f_t \rho(\gamma) f_t^{-1} \quad \text{for } \gamma \in B \\ \rho_t(a) &= \rho(a) \\ \rho_t(b) &= b_t. \end{aligned}$$

This gives a continuous family of well defined representations, with $\rho_1(b)$ the standard order 2 rotation, and $\rho_1(a) \in \mathrm{PSL}(2, \mathbb{R})$.

To finish the proof of the lemma, it suffices to note that, for a sufficiently small deformation b'_t of $\rho_1(b)$ in $\mathrm{SO}(2)$, the commutator $[\rho_1(a), b'_t]$ will remain a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$, as the set of hyperbolic elements is open. Thus, there is a continuous path of conjugacies in $\mathrm{Homeo}^+(S^1)$ to $[\rho_1(a), b]$. This allows us to build a deformation ρ' of ρ with $\rho'(b) = b'_t \in \mathrm{SO}(2)$, using the strategy from Corollary 3.4. Since $\mathrm{rot}(b'_t) \neq \mathrm{rot}(b) = 1/2$, it follows that ρ' and ρ are not semi-conjugate. $\square$

The following corollary summarizes the results of this section.

Corollary 3.10. *Let $\sim_i$ denote the equivalence relation on nonseparating simple closed curves in Σ_g generated by $a \sim_i b$ if $i(a,b) = \pm 1$. Suppose $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ is path-rigid, and suppose that there are simple closed curves a, b with $i(a,b) = \pm 1$ such that $\widetilde{\mathrm{rot}}[\rho(a), \rho(b)] = \pm 1$. Then ρ is semi-conjugate to a (minimal) representation with $\rho(\gamma)$ hyperbolic for all $\gamma \sim_i a$.*

Remark 3.11. In fact, as is proved in [10], the relation $\sim_i$ has only a single equivalence class! However, we will not need to use this fact here, so to keep the proof as self-contained and short as possible we will not refer to it further.

4. Step 3: configuration of fixed points

The objective of this section is to organize the fixed points of the hyperbolic elements in a directed 5-chain; we will achieve this gradually by considering first 2-chains, then 3-chains, and finally 5-chains.

As in Definition 3.1, for a hyperbolic element $f \in \mathrm{Homeo}^+(S^1)$ we let f_+ denote the attracting fixed point of f, and f_- the repelling point. By "$\mathrm{Fix}(f)$ *separates* $\mathrm{Fix}(g)$" we mean that g_- and g_+ lie in different connected components of $S^1 \smallsetminus \mathrm{Fix}(f)$. In particular, $\mathrm{Fix}(f)$ and $\mathrm{Fix}(g)$ are disjoint.

Lemma 4.1. *Let ρ be path-rigid and minimal, and let a, b be simple closed curves with $i(a,b) = \pm 1$ and $\rho(a)$ hyperbolic. Then $\rho(b)$ is hyperbolic, and $\mathrm{Fix}(\rho(a))$ separates $\mathrm{Fix}(\rho(b))$ in S^1.*

Proof. That $\rho(b)$ is hyperbolic follows from Proposition 3.8 above.

As a first step, let us prove that $\mathrm{Fix}(\rho(a))$ and $\mathrm{Fix}(\rho(b))$ are disjoint. Suppose for contradiction that they are not. Then, (after reversing orientations if needed) we have $\rho(a)_+ = \rho(b)_+$. Let I be a neighborhood of $\rho(a)_+$ with closure disjoint from $\{\rho(a)_-, \rho(b)_-\}$. Then, for $N > 0$ large enough, we have $\overline{I} \subset \rho(a^{-N}b)(I)$. Let ρ_t be a bending deformation with $\rho_0 = \rho$, $\rho_t(a) = \rho(a)$ and $\rho_1(b) = \rho(a^{-N}b)$. By Corollary 2.11, $\rho_1(b)$ is hyperbolic. Since $\overline{I} \subset \rho(a^{-N}b)(I)$, its attracting fixed point is outside I, and hence $\rho_1(b)_+ \neq \rho_1(a)_+$. But ρ and ρ_1 are conjugate by Corollary 2.11; this is a contradiction.

Now that we know that $\mathrm{Fix}(\rho(a)) \cap \mathrm{Fix}(\rho(b)) = \emptyset$, we will prove that they separate each other. Suppose for contradiction that $\mathrm{Fix}(\rho(a))$ does not separate $\mathrm{Fix}(\rho(b))$. Up to conjugating ρ by an orientation-reversing homeomorphism of S^1, and up to replacing b with b^{-1}, the fixed points of $\rho(a)$ and $\rho(b)$ have cyclic order (a_+, a_-, b_+, b_-). (For simplicity, we have suppressed the notation ρ.)

Fix $N \in \mathbb{N}$ large, and let ρ' be a bending deformation of ρ so that $\rho'(b) = \rho(a^N)\rho(b)$, and $\rho'(a) = \rho(a)$. It follows from Corollaries 2.11 and 3.7 that, if N is large enough, the points $b'_+ = \rho'(b)_+$ and $b'_- = \rho'(b)_-$ can be taken arbitrarily close, respectively, to a_+ and $\rho(b)^{-1}(a_-)$. Since the cyclic order of fixed points is preserved under deformation they are also in order (a_+, a_-, b'_+, b'_-). This is incompatible with the positions of a_+ and $\rho(b)^{-1}(a_-)$, unless perhaps if $\rho(b)^{-1}(a_-) = a_+$. But if $\rho(b)^{-1}(a_-) = a_+$, then $\rho'(b)$ has no fixed point in $(\rho(b)^{-1}(a_+), a_+)$ as this interval is mapped into (a_+, a_-) by $\rho(b')$. This again gives an incompatibility with the cyclic order. $\square$

Lemma 4.2. *Let ρ be path-rigid and minimal, and let (a, b, c) be a directed 3-chain. Suppose that $\rho(a)$ is hyperbolic, and suppose that $\rho(a)$ and $\rho(c)$ do not have a common fixed point. Then $\rho(b)$ and $\rho(c)$ are hyperbolic, and, up to reversing the orientation of S^1, their fixed points are in the cyclic order*

$$(\rho(a)_-, \rho(b)_-, \rho(a)_+, \rho(c)_-, \rho(b)_+, \rho(c)_+).$$

Proof. It follows from Proposition 3.8 that $\rho(b)$ and $\rho(c)$ are hyperbolic, and from Lemma 4.1 that up to reversing orientation, the fixed points of $\rho(a)$ and $\rho(b)$ come in the cyclic order

$$(a_-, b_-, a_+, b_+).$$

(For simplicity we drop ρ from the notation for the fixed points of a, b and c). As mentioned above, the effect of a bending deformation that realizes a power of a Dehn twist along a is to leave a and c invariant and to replace b with ba^N. Corollary 2.11 says that the resulting representation is conjugate to ρ. By doing this with $N > 0$ and $N < 0$ large, we get representations for which $b'_- = \rho(ba^N)_-$ can be taken arbitrarily close to a_+, as well as to a_-. This, and Lemma 4.1 applied to the curves (b, c), imply that the intervals (a_+, b_+) and (b_+, a_-) each contain one fixed point of c. In order to prove the lemma, it now suffices to prove that the cyclic order of fixed points

$$(a_-, b_-, a_+, c_+, b_+, c_-)$$

cannot occur. Suppose for contradiction that this configuration holds, and apply a power of Dehn twist along b, replacing a with $b^{-N}a$ and c with cb^N (and leaving b invariant), for $N > 0$ large. Denote by c'_+, c'_-, a'_- and a'_+ the resulting fixed points, ie, the fixed points of $\rho(cb^N)$ and $\rho(b^{-N}a)$ for $N > 0$ large. If N is chosen large enough, then c'_+, c'_- and a'_- are arbitrarily close to $c(b_+)$, b_- and $a^{-1}(b_+)$ respectively. (See Corollary 3.7 above.) These three points are in the inverse cyclic order as c_+, c_- and a_-, hence, the representation ρ' obtained from this Dehn twist cannot be conjugate to ρ. This contradicts Corollary 2.11, so eliminates the undesirable configuration. □

We are now ready to prove the main result of this section.

Proposition 4.3. *Let ρ be a path-rigid, minimal representation, and let (a, b, c, d, e) be a directed 5-chain in Σ_g. Suppose $\rho(a)$ is hyperbolic. Then, $\rho(b),\dots,\rho(e)$ are hyperbolic as well, and up to reversing the orientation of the circle, their fixed points are in the following (total) cyclic order:*

$$(a_-, b_-, a_+, c_-, b_+, d_-, c_+, e_-, d_+, e_+).$$

In particular, these fixed points are all distinct. As before, for simplicity we have dropped ρ from the notation.

Proof. That $\rho(b),\dots,\rho(e)$ are all hyperbolic follows from Proposition 3.8. Next, using a bending deformation realizing a Dehn twist along d, we may change the action of c into $d^{-N}c$ without changing a, and without changing the conjugacy class of ρ. In particular, such a deformation moves the fixed points of c, so we can ensure that $\mathrm{Fix}(\rho(a))$ and $\mathrm{Fix}(\rho(c))$ are disjoint.

Similarly, for any two elements in the chain (a, b, c, d, e), there is a third one that intersects one but not the other. Thus, we may apply the same reasoning to show that all these five hyperbolic elements have pairwise disjoint fixed sets. It remains to order these fixed sets. For this, we will apply Lemma 4.2 repeatedly.

First, fix the orientation of S^1 so that, applying Lemma 4.2 to the directed 3-chain (a, b, c), we have the cyclic order of fixed points

$$(a_-, b_-, a_+, c_-, b_+, c_+).$$

Now, Lemma 4.2 applied to the directed 3-chain (b, c, d) implies that d_- lies in the interval (b_+, c_+) and d_+ in the interval (c_+, b_-). Applying the lemma to the directed 3-chain (a, cb, d) implies that d_+ in fact lies in the interval (c_+, a_-).

The same argument using Lemma 4.2 applied to the directed 3-chains (c, d, e) and (a, dcb, e) shows that e_- lies in the interval (c_+, d_+) and e_+ in the interval (d_+, a_-), as desired. ☐

5. Step 3: Maximality of the Euler number

In order to compute the Euler number of ρ, we will decompose Σ_g into subsurfaces and compute the contribution to $\mathrm{eu}(\rho)$ from each part. The proper framework for discussing this is the language of bounded cohomology: if Σ is a surface with boundary $\partial\Sigma$, and $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}^+(S^1)$, one obtains a characteristic number by pulling back the *bounded Euler class* in $H^2_b(\mathrm{Homeo}^+(S^1); \mathbb{R})$ to $H^2_b(\Sigma; \mathbb{R}) \cong H^2_b(\Sigma, \partial\Sigma; \mathbb{R})$ and pairing it with the fundamental class $[\Sigma, \partial\Sigma]$. The contribution to the Euler number of $\rho : \Sigma_g \to \mathrm{Homeo}^+(S^1)$ from a subsurface Σ is simply this Euler number for the restriction of ρ to Σ.

However, in order to keep this work self-contained and elementary, we will avoid introducing the language of bounded cohomology, and give definitions in terms of rotation numbers alone. The reader may refer to [1, § 4.3] for details on the cohomological framework.

Definition 5.1 (Euler number for pants)**.** Let $\rho : \Gamma_g \to \mathrm{Homeo}^+(S^1)$, and let $P \subset \Sigma_g$ be a subsurface homeomorphic to a pair of pants, bounded by curves a, d and $(da)^{-1}$, with orientation induced from the boundary. Let $\widetilde{\rho(a)}$ and $\widetilde{\rho(d)}$ be any lifts of $\rho(a)$ and $\rho(d)$ to $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. The *Euler number of ρ on P* is the real number

$$\mathrm{eu}_P(\rho) = \widetilde{\mathrm{rot}}\left(\widetilde{\rho(a)}\right) + \widetilde{\mathrm{rot}}\left(\widetilde{\rho(b)}\right) - \widetilde{\mathrm{rot}}\left(\widetilde{\rho(d)}\widetilde{\rho(a)}\right).$$

An illustration in the case where P contains the basepoint is given in Figure 2.

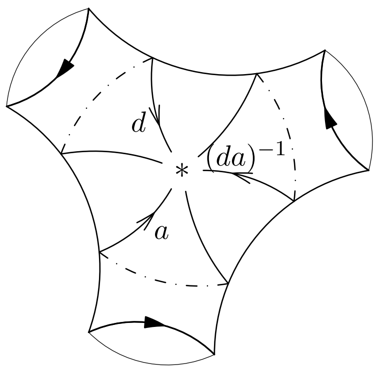

Figure 2. A pair of pants with standard generators of its fundamental group

Note that the number $\mathrm{eu}_P(\rho)$ is independent of the choice of lifts of $\rho(a)$ and $\rho(d)$. We also allow for the possibility that the image of P in Σ_g has two boundary curves identified, so is a one-holed torus subsurface. We may choose free generators a, b for the torus, with $i(a, b) = -1$ so the torus is $T(a, b)$ and the boundary of P is given by the curves $b^{-1}, a^{-1}ba$ and the commutator $[a,b]$. Then the definition above gives $\mathrm{eu}_P(\rho) = \widetilde{\mathrm{rot}}[\widetilde{\rho(a)}, \widetilde{\rho(b)}]$.

Lemma 5.2. *Let P be any pants and ρ a representation. Then $|\mathrm{eu}_P(\rho)| \leq 1$.*

A proof using the language of rotation numbers (consistent with our notation) can be found in [2, Theorem 3.9]. However this bound is classical and was known much earlier. For example, the case for one-holed torus subsurfaces appears in [15, Prop. 4.8], and the general case is implicit in [3].

More generally, if $S \subset \Sigma_g$ is any subsurface, we define the Euler number $\mathrm{eu}_S(\rho)$ to be the sum of relative Euler numbers over all pants in a pants decomposition of S. From the perspective of bounded cohomology, it is immediate that this sum does not depend on the pants decomposition; however, since we are intentionally avoiding cohomological language, we give a short stand-alone proof.

Lemma 5.3. *For any subsurface $S \subseteq \Sigma_g$, the number $\mathrm{eu}_S(\rho)$ is well-defined, i.e. independent of the decomposition of S into pants.*

Proof. Any two pants decompositions can be joined by a sequence of elementary moves; namely those of type (I) and (IV) as shown in [7]. A type (IV) move takes place within a pants-decomposed one-holed torus P so does not change the value of eu_P, which is simply the rotation number of the boundary curve, as remarked above. Thus, the move does not change the sum of relative Euler numbers. A type (I) move occurs within a four-holed sphere S'; if the boundary of the sphere is given by oriented curves a, b, c, d with $dcba = 1$, then it consists of replacing the decomposition along da with a decomposition along ab. It is easy to verify by the definition that, in either case, the sum of the Euler numbers of the two pants on S' is given by $\widetilde{\mathrm{rot}}\left(\widetilde{\rho(a)}\right) + \widetilde{\mathrm{rot}}\left(\widetilde{\rho(b)}\right) + \widetilde{\mathrm{rot}}\left(\widetilde{\rho(c)}\right) + \widetilde{\mathrm{rot}}\left(\widetilde{\rho(d)}\right)$. □

Proposition 5.4 (Additivity of Euler number)**.** *Let $\mathcal{P}$ be any decomposition of Σ into pants. Then $\mathrm{eu}(\rho) = \sum_{P \in \mathcal{P}} \mathrm{eu}_P(\rho)$.*

By Lemma 5.3, we may use any pants decomposition to compute the Euler class. By using a standard generating system $(a_1, \dots, b_g)$ and cutting Σ_g along geodesics freely homotopic to a_i, $c_i = [a_i, b_i]$, for $i = 1, \dots, g$ and $d_i = c_i \cdots c_1$ for $i = 2, \dots, g-1$, we recover the formula taken as a definition in Proposition 2.2.

We now return to our main goal: we prove that maximality of the Euler class holds first on small subsurfaces, then globally on Σ_g.

Proposition 5.5. *Let $S \subset \Sigma_g$ be a subsurface homeomorphic to a four-holed sphere. Suppose that none of the boundary components of S is separating in Σ_g, and let ρ be a path-rigid, minimal representation mapping one boundary component of S to a hyperbolic element of $\mathrm{Homeo}^+(S^1)$. Then, ρ maps all four boundary components of S to hyperbolic elements, and the relative Euler class $\mathrm{eu}_S(\rho)$ is equal to ± 2.*

In the statement above, we do not require that the boundary components are geodesics for some metric on Σ_g, in particular, two of them may well be freely homotopic.

Proof. Put the base point inside of S. The complement $\Sigma_g \smallsetminus S$ may have one or two connected components, since none of the curves of ∂S are separating in Σ_g. In either case, we may find two based, nonseparating, simple closed curves $u, v \in \Gamma_g$, with $i(u,v) = 0$, each having nonzero intersection number with exactly two of the boundary components of S, as shown in Figure 3. Additionally, we may fix orientations for u and v and choose four elements $a, b, c, d \in \pi_1 S$, each freely homotopic to a different boundary component of S, with $dcba = 1$, and such that $(a, u, d^{-1}a^{-1}, v, d)$ and (c, v, ad, u, b) are directed 5-chains in Σ_g. As we have

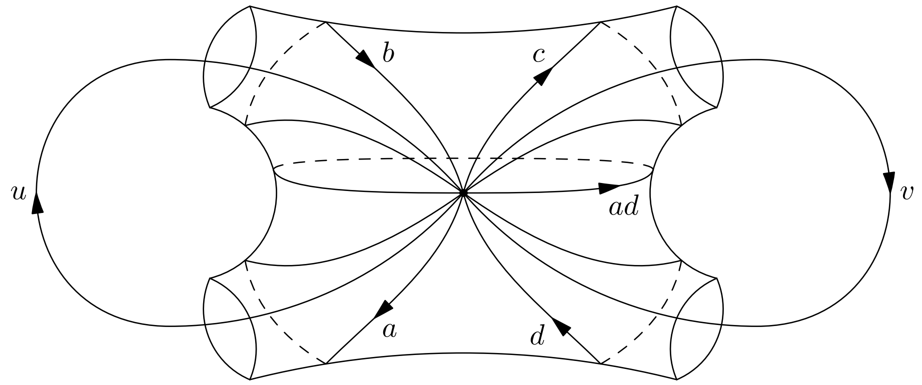

FIGURE 3. A four-holed sphere and two 5-chains

assumed that the image under ρ of one of a, b, c or d is hyperbolic, Proposition 3.8 implies that all the curves appearing in these 5-chains are in fact hyperbolic.

Orient the circle so that $(u_-, (ad)_+, u_+, (ad)_-)$ are in cyclic order (as before, we drop the letter ρ from the notation, for better readability). Then, Proposition 4.3 applied to the two directed 5 chains above gives the cyclic orderings

$$(a_-, u_-, a_+, (ad)_+, u_+, v_-, (ad)_-, d_-, v_+, d_+)$$

and

$$(c_-, v_-, c_+, (ad)_-, v_+, u_-, (ad)_+, b_-, u_+, b_+).$$

These two orderings together yield the cyclic ordering

$$((ad)_-, d_-, d_+, a_-, a_+, (ad)_+, b_-, b_+, c_-, c_+).$$

We now use this ordering to prove maximality of the Euler class. Let α, β, γ and δ, respectively, denote the lifts of $\rho(a)$, $\rho(b)$, $\rho(c)$ and $\rho(d)$ to $\mathrm{Homeo}_{\mathbb{Z}}^+(\mathbb{R})$ with translation number zero. Let $x = (ad)_-$ be the repelling fixed point of ad.

Since x has a repelling fixed point of d immediately to the right, and an attracting fixed point of d to the left, we have $\delta(x) < x$. By the same reasoning, if y is any point in the interval between consecutive lifts of fixed points a_+ and a_- containing x, then $\alpha(y) < y$. Since $ad(x) = x$, it follows that $\delta(x)$ must lie to the left of the lift of a_+, and we have $\alpha\delta(x) = x - 1$.

Since $cbad = 1$, we also have that $cb(x) = x$. Considering the location of repelling points of b and c and imitating the argument above, we have again $\beta(x) < x$, and also $\gamma\beta(x) < x$. It follows that $\gamma\beta(x) = x - 1$, hence $\gamma\beta\alpha\delta(x) = x - 2$, and $\mathrm{eu}_S(\rho) = -2$. □

Using this information about subsurfaces, we can prove that the Euler number of ρ is maximal.

Proposition 5.6. *Let ρ be path-rigid, and suppose that ρ admits a Fuchsian torus. Then ρ has Euler number $\pm(2g-2)$.*

Proof. After semi-conjugacy, we may assume that ρ is minimal. Let $T(a, b)$ be a Fuchsian torus for ρ. By Corollary 3.4, we may suppose that $\rho(a)$ is hyperbolic. Ignoring the curve b, find a system of simple closed curves $a_1 = a, a_2, \ldots, a_{g-1}$, with each a_i nonseparating, that decomposes Σ_g into a disjoint union of pairs of pants.

The dual graph of such a pants decomposition is connected (because Σ_g is connected), so we may choose a finite path that visits all the vertices. In other words, we may choose a sequence $P_1, \dots, P_N$ of pants from the decomposition (possibly with repetitions), that contains each of the pants of the decomposition, such that each two consecutive pants P_i and P_{i+1} are distinct, but share a boundary component. Let S_i denote the four-holed sphere obtained by taking the union of P_i and P_{i+1} along a shared boundary curve. (If P_i and P_{i+1} share more than one boundary component, choose only one). We may further assume that a is one of the boundary curves of S_1.

Starting with S_1 as the base case, and applying Proposition 5.5, we inductively conclude that all boundary components of all the S_i are hyperbolic, and that $\mathrm{eu}_{S_i}(\rho) = \pm 2$. Thus, the contributions of P_i and P_{i+1} are equal, and equal to ± 1, for all i. It follows that the contributions of all pairs of pants of the decomposition have equal contributions, equal to ± 1. By definition of the Euler class, we conclude that $\mathrm{eu}(\rho) = \pm(2g-2)$. □

The proof of Theorem 1.3 now concludes by citing Matsumoto's result of [11] that such a representation of maximal Euler number is semi-conjugate to a Fuchsian representation. □

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