ON FACTORING HOMEOMORPHISMS

The following proposition makes a necessary improvement and small correction to statements in [LM18] and [Man24]. The application to these two papers is explained after the proof.

Proposition 0.1. *Let M_0 be an open compact manifold diffeomorphic to the interior of a compact manifold M with boundary. Let $N \cong (0,\infty) \times \partial M$ be the complement of a (sufficiently large) compact subset of M_0. Then:*

1. *Given a C^r diffeomorphism f of M_0 isotopic to the identity, there exists a factorization $f = kgh$ where k is compactly supported, and g and h are supported on sets of the form $X = \sqcup_{i\in\mathbb{N}}[a_i, b_i] \times \partial M$, and $Y = \sqcup_{i\in\mathbb{N}}[c_i, d_i] \times \partial M$ in N, where $a_i < b_i < c_i < d_i < a_{i+1}$.*
2. *Let $X, Y \subset N$ be sets as above. If g, h are any C^r diffeomorphisms supported on X and Y respectively; then there exist diffeomorphisms g' and h' supported on X and Y (respectively) which are isotopic to the identity rel ∂X and rel ∂Y (respectively) such that $gh = g'h'$.*
3. *Furthermore, if g' and h' are as above, and lie in the commutator subgroup of $\mathrm{Diff}^r_0(X)$ and $\mathrm{Diff}^r_0(Y)$ respectively, then each can be written as a product of two commutators, and so $g'h'$ has commutator length 4.*

Here we allow any value of r including 0, which is the case of homeomorphisms. In particular, if $r \neq \dim(M) + 1$, then $\mathrm{Diff}^r_0(X)$ and $\mathrm{Diff}^r_0(Y)$ are simple groups so every element is a commutator and g' and h' have commutator length 2 in $\mathrm{Diff}^r_0(X)$ and $\mathrm{Diff}^r_0(Y)$ respectively.

Proof. The factorization of $f = kgh$ as in item (1) is done in the proof of Proposition 5.1 of [LM18]. (In fact, there it is shown that given a countable collection f_n, there is a single choice of set X and Y as above where one can produce a factorization $k_n g_n h_n$ with g_n supported on X and h_n supported on Y.)

What remains to prove is items 2 and 3.

To simplify notation, let $X_i = [a_i, b_i] \times \partial M$, and $Y_i = [c_i, d_i] \times \partial M$. Let A_i denote $X_i \cap Y_i$ and $B_i = Y_i \cap X_{i+1}$.

We will modify g and h, not changing their support, so that they are isotopic to the identity rel boundary of X_i and Y_i (respectively) and so that their support and their product gh remains unchanged.

To do this, let g_i denote the restriction of g to X_i. Let $\{d_i\}$, $\{e_i\}$ be two sequences of C^r diffeomorphisms, with d_i supported on A_i and e_i is supported on B_i, defined inductively as follows. Let $e_0 = id$, and choose d_1 so that $g_1 d_1 e_0^{-1}$ is isotopic to identity rel boundary on X_1, and let $g'_1 = g_1 d_1 e_0^{-1}$. Now choose e_1 so that $d_1^{-1} e_1 h_1$ is isotopic to identity rel boundary on Y_1, and denote this map by h'_2. Continue iteratively, defining d_i so that $g'_i := g_i d_i e_{i-1}^{-1}$ is isotopic to identity rel boundary on X_i and $h'_i := d_i^{-1} e_i h_i$ is isotopic to identity rel boundary on Y_i. Let g' be the product of all g'_i (these have disjoint support) and h' the product of all h'_i.

We claim that $gh = g'h'$. To see this, one checks cases: if $x \in h^{-1}A_i$ then $h_i(x) \in A_i$ where all e_j are identity, so $g'h'(x) = g_i h_i(x) = gh(x)$. Similarly, if $x \in h^{-1}B_i$, the maps d_j are identity and the e_i factors cancel out so $g'h'(x) = g_i h_i(x) = gh(x)$. In the remaining case, $h(x)$ lies in a set where all d_j and e_j are identity so again we have $g'h'(x) = gh(x)$. This proves item (2).

For item (3) we use the fact that X_i and Y_i are *portable manifolds* in the sense of [BIP08]: any compact subset of the interior of X_i can be displaced (infinitely many times) to be all pairwise disjoint from each other. The proof of Theorem 2.2(i) in [BIP08] paper shows that any commutator g_i (or h_i) in the diffeomorphism (or homeomorphism) group of such a manifold can be written as a product of two commutators. $\square$

Addenda to [LM18] and [Man24]. Lemma 4.4 (a special case) and Proposition 5.1 (generally) of [LM18] construct factorizations $f = kgh$ as in Proposition 0.1, however later on it is assumed that g and h can be taken isotopic to identity rel ∂X and ∂Y respectively. This step is not justified in [LM18], but is now justified given item (2) of Proposition 0.1.

In section 4 of [Man24], the proof of automatic continuity for Homeo(M), with M noncompact, without marked points, begins with a incorrect sketch of a factorization argument, writing a homeomorphism of M as a product $f = kh_1h_2$ (where h_1 and h_2 play the roles of g' and h' above). This sketch can be replaced by Proposition 0.1 above, and the (incorrect) assertion that h_i can be written as a commutator should be replaced by statement (3) of Proposition 0.1, so it is a product of two commutators. This does not affect the statement of the main theorem.

References

[BIP08] Dmitri Burago, Sergei Ivanov, and Leonid Polterovich. Conjugation-invariant norms on groups of geometric origin. In *Groups of diffeomorphisms*, volume 52 of *Adv. Stud. Pure Math.*, pages 221–250. Math. Soc. Japan, Tokyo, 2008.

[LM18] F. Le Roux, K. Mann, *Strong distortion in transformation groups.* Bulletin of the London Math. Soc. 50.1 (2018), 46-62.

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