

ROTATION SETS AND ACTIONS ON CURVES

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ABSTRACT. Building on work of [BHW21], we study the action of the homeomorphism group of a surface S on the fine curve graph $\mathcal{C}^\dagger(S)$. While the definition of $\mathcal{C}^\dagger(S)$ parallels the classical curve graph for mapping class groups, we show that the dynamics of the action of $\text{Homeo}(S)$ on $\mathcal{C}^\dagger(S)$ is much richer: homeomorphisms induce parabolic isometries in addition to elliptics and hyperbolics, and all positive reals are realized as asymptotic translation lengths.

When the surface S is a torus, we relate the dynamics of the action of a homeomorphism on $\mathcal{C}^\dagger(S)$ to the dynamics of its action on the torus via the classical theory of *rotation sets*. We characterize homeomorphisms acting hyperbolically, show asymptotic translation length provides a lower bound for the area of the rotation set, and, while no characterisation purely in terms of rotation sets is possible, we give sufficient conditions for elements to be elliptic or parabolic.

1. INTRODUCTION

For a closed surface S of genus $g \geq 1$, the classical *curve graph* $\mathcal{C}(S)$ has vertex set the isotopy classes of essential simple closed curves on S , with edges between pairs of isotopy classes that can be realized disjointly (a slight modification is needed for genus 1 surfaces). Masur and Minsky [MM99, MM00] showed that the graph $\mathcal{C}(S)$ is Gromov hyperbolic, a result which has become an essential tool in the study of the geometric and algebraic structure of the mapping class group $\text{Map}(S)$, see for example [BM08, Mah11, BKMM12, BBF15, DGO17].

In [BHW21] three of the authors introduced the *fine curve graph* $\mathcal{C}^\dagger(S)$ to study the group of all homeomorphisms of S . This graph has essential simple closed curves as vertices, so admits a faithful action of $\text{Homeo}(S)$ by isometries. It is shown in [BHW21] that $\mathcal{C}^\dagger(S)$ has infinite diameter and is hyperbolic (here, the assumption on positive genus is necessary, see [BHW21, Section 5.2]). This enables large scale geometric techniques for studying $\text{Homeo}(S)$ via its action on $\mathcal{C}^\dagger(S)$, for instance, stable commutator length and fragmentation norm on $\text{Homeo}_0(S)$ are unbounded, answering a question posed by Burago, Ivanov, and Polterovich [BIP08].

In this paper, we show that there is a rich correspondence between the dynamics of the induced action of a homeomorphism on $\mathcal{C}^\dagger(S)$ and its dynamics on the surface S itself. We first address this in a general setting, then

specialize to the case of the torus where we study the interactions between the curve graph and the existing dynamical theory of *rotation sets* for torus homeomorphisms.

1.1. General results. Isometries of hyperbolic metric spaces admit a dynamical trichotomy as *elliptic*, *parabolic* or *hyperbolic* according to the asymptotic translation length and diameter of orbits (see Section 2 for a review of definitions). In the classical setting of $\mathcal{C}(S)$, it follows from [MM99, Proposition 4.6] and the Nielsen–Thurston classification [Thu88] that no mapping classes act parabolically. By contrast, we show the following.

Theorem 1.1 (Parabolic examples). *For any closed orientable surface $S \neq S^2$ there exist isotopically trivial homeomorphisms of S whose action on $\mathcal{C}^\dagger(S)$ is parabolic.*

The proof of Theorem 1.1 is via explicit constructions, some of which also yield smooth examples. Hyperbolic and elliptic isometries are much easier to build: for the elliptic case, it is easy to define many homeomorphisms which fix a given curve, and many examples of hyperbolics are given in [BHW21].

Our next result shows that the action of $\text{Homeo}(S)$ on $\mathcal{C}^\dagger(S)$ is dynamically very rich; it may be interpreted as giving some justification of the pervasiveness of hyperbolic examples.

Theorem 1.2 (Continuity). *Let S have genus $g \geq 1$. Asymptotic translation length on $\mathcal{C}^\dagger(S)$ is a continuous function on $\text{Homeo}(S)$. Consequently,*

- (1) *all nonnegative real numbers can be realized as asymptotic translation lengths in $\text{Homeo}_0(S)$, and*
- (2) *hyperbolicity is an open condition in the C^0 topology.*

Interestingly, while the set of elliptic mapping classes is not open, one can easily construct examples of open subsets of $\text{Homeo}(S)$ consisting entirely of elliptics, see Construction 3.3, which can be taken to be smooth. Hence hyperbolicity is not generic in the C^0 topology.

We highlight another key difference to the classical setting of $\mathcal{C}(S)$ via Theorem 1.2. Indeed, Bowditch [Bow08] proved that asymptotic translation lengths on $\mathcal{C}(S)$ belong to $\frac{1}{m}\mathbb{Z}$ with m depending only on S i.e. they are uniformly rational. On the other hand Theorem 1.2 shows that all nonnegative reals occur as asymptotic translation lengths on $\mathcal{C}^\dagger(S)$.

1.2. Torus homeomorphisms and rotation sets. In the case of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$, there is a well-developed theory of the dynamics of isotopically trivial homeomorphisms via their *rotation sets*, subsets of \mathbb{R}^2 which, loosely speaking, measure the average displacement of points under iteration. The influential paper [MZ89] of Misiurewicz and Ziemian sparked a general program to relate the dynamics of torus homeomorphisms to the geometric and topological properties of their rotation sets. Their work shows that rotation sets are compact and convex, so these are either points, line segments or have nonempty interior. Each of these do in fact arise: by considering

homeomorphisms of the torus which preserve a foliation by circles and act like a rotation on each of those circles, it is easy to produce examples of homeomorphisms whose rotation set is a singleton or a segment of rational slope containing rational points. Homeomorphisms whose rotation set is a segment with irrational slope were constructed by Katok (see [Han89, Example 1.4] and [Kwa07]); while Avila announced a construction of an example whose rotation set is a segment contained in a line of irrational slope and with no rational point. Le Calvez and Tal [LCT18] introduced a new orbit forcing theory and proved, among other results, that an irrational slope line rotation set cannot have a rational point in its interior, which verifies a case of the Franks–Misiurewicz conjecture [FM90].

We show that the topology of the rotation set classifies homeomorphisms of the torus which act hyperbolically on $\mathcal{C}^\dagger(T)$:

Theorem 1.3 (Hyperbolic characterisation, special case). *Let $f \in \text{Homeo}_0(T)$. The following are equivalent*

- (1) *f acts hyperbolically on $\mathcal{C}^\dagger(T)$,*
- (2) *$\rho(f)$ has non-empty interior, and*
- (3) *there is a finite, f -invariant set $P \subset T$ such that the restriction of f to $T - P$ represents a pseudo-Anosov mapping class.*

We also show that the area of the rotation set is bounded from below in terms of the square of the asymptotic translation length, see Proposition 5.2.

The above theorem highlights a new connection between *geometry* of $\mathcal{C}^\dagger(S)$, *dynamics* via the rotation set, and *topology* via the mapping class represented in the complement of an invariant finite set of points. We note that the implication from (2) to (3) above was already known: it is a consequence of the work of Franks [Fra89] and Llibre–MacKay [LM91]. The implication from (3) to (2) is more subtle because the rotation vectors of P may be equal. Nonetheless Boyland [Boy09] shows that the Thurston representative on $T - P$ has rotation set with non-empty interior, and so (2) can be deduced from Nielsen fixed point theory and the convexity of the rotation set. In this paper we prove (1) implies (2) and (3) implies (1) using different methods.

Following [Doe97], one may also define a rotation set for homeomorphisms of T that admit a power isotopic to a (power of a) Dehn twist map. This set measures the speed of orbits transversally to the twist; see Definition 2.14. Using this, we extend Theorem 1.3 to give an analogous characterisation applicable to all homeomorphisms of T . The precise statement is given in Theorem 5.3 below.

One might hope that elliptic and parabolic isometries of $\mathcal{C}^\dagger(T)$ could similarly be distinguished by their rotation sets. This is half-true: we show that one can give sufficient conditions separately for parabolicity and ellipticity, but also show via explicit examples that one cannot hope to give both sufficient and necessary conditions as in Theorem 1.3.

Theorem 1.4. *Let $f \in \text{Homeo}_0(T)$. If $\rho(f)$ is a segment of irrational slope, then f acts on $\mathcal{C}^\dagger(T)$ parabolically. If $\rho(f)$ is a segment of rational slope containing rational points, then f acts elliptically.*

Note that there are many elliptic elements whose rotation set is a singleton e.g. any homeomorphism whose support is contained in a disk has null rotation set. Thus, the condition on elliptics in Theorem 1.4 is not necessary. For parabolics, in Section 6 we also show

Proposition 1.5. *There are homeomorphisms f that act parabolically on $\mathcal{C}^\dagger(T)$ with $\rho(f) = \{(0, 0)\}$.*

This implies, in particular, that one cannot hope to distinguish elliptics from parabolics via rotation sets alone. Some of our examples can be chosen to be smooth.

1.3. Further questions. An interesting related question is to study actions on analogous graphs for infinite type surfaces, as a kind of intermediate point between $\mathcal{C}^\dagger(S)$ and $\mathcal{C}(S)$. Work of Bavard [Bav16] and of Horbez–Qing–Rafi [HQR21] shows that many of these groups admit dynamically interesting actions on hyperbolic metric spaces. However, there is no known analog of the Nielsen–Thurston classification, and whether these actions have parabolic elements appears to be open.

It is natural to try to extend Theorem 1.3 on rotation sets for tori to the case of higher genus surfaces. We conjecture that the equivalence of (1) and (3) should hold generally. However, existing analogs of the rotation set on higher genus surfaces do not lend themselves as easily to such an analysis as we do here. Guihéneuf and the fourth author [GM22] recently introduced the notion of *homotopic rotation sets*, and proved that the desired periodic points exist under slightly stronger conditions on the corresponding rotation sets, giving a step towards the equivalence of statements analogous to (2) and (3).

There is also the question of the relationship between the asymptotic translation length on the fine curve graph, and the topological entropy of the homeomorphism. Since homeomorphisms supported on a disc can have arbitrarily large topological entropy, there can be no upper bound on entropy in terms of the asymptotic translation length. On the other hand, a lower bound seems to be within reach. For the torus, the area of the rotation set is bounded from below in terms of the asymptotic translation length, see Proposition 5.2 below. The work of Kwapisz [Kwa95] provides lower bounds on the topological entropy in terms of shape and area of the rotation set.

1.4. Outline. The paper is organised as follows. In Section 2 we briefly provide the necessary background for our theorems and proofs. In Section 3 we show that the asymptotic translation length of a homeomorphism on $\mathcal{C}^\dagger(S)$ is C^0 -continuous, and therefore C^r -continuous for $1 \leq r \leq \infty$. In Section 4 we provide useful bounds on distances in $\mathcal{C}^\dagger(S)$, which enable us

to give statements about rotation sets given actions on curves, or vice versa. These are of independent interest, but also used in our later proofs. In Section 5 we provide the proofs of Theorems 1.3 and 5.3. In Section 6 we provide the proofs of Theorem 1.1 and Proposition 1.5.

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2. BACKGROUND

In this section, we set up notation and recall basic facts from coarse geometry and topological dynamics to be used later in this work.

2.1. Hyperbolic spaces and their isometries. Let X be a geodesic metric space.

Notation 2.1. For points x, y in X , we denote by $[x, y]$ any geodesic between them. We make the convention that statements about $[x, y]$ are supposed to hold for any choice of geodesic.

We say that X is δ -hyperbolic if geodesic triangles are δ -*slim* i.e. if for all points $x, y, z \in Z$ we have

$$[x, y] \subset N_\delta([y, z] \cup [z, x]).$$

Here, N_δ denotes the closed δ -neighbourhood. We say that X is *Gromov hyperbolic* (or just *hyperbolic*) if X is δ -hyperbolic for some $\delta \geq 0$. For details, background, and basic properties of Gromov hyperbolic spaces, we refer the reader to [BH99, Chapter III.H]. It is a straightforward consequence of the definition that the Hausdorff distance between any geodesics joining the same two points is bounded by δ .

For an isometry g of a Gromov hyperbolic space X , the *asymptotic translation length* is defined as

$$|g|_X := \lim_{n \rightarrow \infty} \frac{1}{n} d_X(x, g^n(x))$$

It is a standard exercise to see that this limit exists and is independent of x . This independence immediately implies that the asymptotic translation length is a conjugacy invariant of isometries of X .

We have the following classification of isometries [Gro87, § 8]:

Definition 2.2. Let g be an isometry of a Gromov hyperbolic space. We say that g is

Hyperbolic: if the asymptotic translation length is positive,

Parabolic: if the asymptotic translation length is zero but g has no finite diameter orbits, and

Elliptic: if g has finite diameter orbits.

Note again that these categories are invariant under conjugation by isometries. The reader may recall that there is an equivalent reformulation of this trichotomy in terms of fixed points on the Gromov boundary of X , but we do not require this point of view in the present work.

2.2. Surfaces and Curves. Throughout this article, *surface* will mean a closed, oriented surface S of genus at least 1, with T being used to denote the torus. A *curve* on such a surface will always be required to be an essential closed loop (i.e. non-trivial in $\pi_1(S)$), frequently we restrict our attention to simple (i.e. topologically embedded) curves.

The main geometric tool we use in this article is the following variant of the classical curve graph, which is the 1-skeleton of the curve complex, see [MM99, Section 2.2] for a definition.

Definition 2.3. For a surface S of genus at least two, the *fine curve graph*, denoted by $\mathcal{C}^\dagger(S)$, is the graph with vertex set equal to the set of essential simple closed curves in S , and edges given by disjointness.

In the case of the torus T , vertices of $\mathcal{C}^\dagger(T)$ are again essential simple closed curves, but two vertices are now joined by an edge when the corresponding curves are either disjoint or intersect topologically transversely at most once.

We endow $\mathcal{C}^\dagger(S)$ with a metric in which each edge is isometric to the unit interval, and the distance between two vertices is given by the minimal length of a path between them. The distance between two vertices α and β of $\mathcal{C}^\dagger(S)$ will be denoted by $d^\dagger(\alpha, \beta)$.

We need two facts about $\mathcal{C}^\dagger(S)$ which are true for any surface S of genus at least 1 (compare also [BHW21, Section 5.2] for comments on the torus case):

Theorem 2.4 ([BHW21, Theorem 3.8]). *The graph $\mathcal{C}^\dagger(S)$ is Gromov hyperbolic.*

Theorem 2.5 ([BHW21, Lemma 4.2]). *Suppose that $F: S \rightarrow S$ is a homeomorphism fixing a finite set $P \subset S$, and let φ be the mapping class of $S - P$ defined by F . Then,*

$$|F|_{\mathcal{C}^\dagger(S)} \geq |\varphi|_{\mathcal{C}(S-P)}.$$

By the work of Masur–Minsky (see [MM99, Proposition 4.6]), if φ is pseudo-Anosov, then $|\varphi|_{\mathcal{C}(S-P)} > 0$ hence F acts hyperbolically on $\mathcal{C}^\dagger(S)$.

Remark 2.6 (Smooth versus non-smooth). In [BHW21], the graph $\mathcal{C}^\dagger(S)$ has vertices corresponding to smooth curves, whereas for the applications here we need to allow all essential curves as vertices. As discussed in [BHW21, Remark 3.2], these two graphs are quasi-isometric, which yields

Theorem 2.4. The argument given in [BHW21] for Theorem 2.5 is stated in the smooth context but applies equally well in the C^0 setting, giving the statement above.

In addition, we use the following easy lemma:

Lemma 2.7. *Suppose that S is any surface of genus $g \geq 1$, and suppose that α and β are two curves on S intersecting in a finite number of points. Then*

$$d^\dagger(\alpha, \beta) \leq 2\#(\alpha \cap \beta) + 2$$

This lemma can be deduced easily from Lemma 3.4 of [BHW21] and the corresponding estimate for usual curve graphs. We give an alternative proof using *curve surgery*, as a warm-up for later arguments which will use similar tools.

Proof. Suppose that α and β are two curves which intersect in a finite number of points. Let $a \subset \alpha$ be an embedded subarc such that $a \cap \beta$ is equal to the endpoints of a . Denote by $b', b'' \subset \beta$ the two connected components of $\beta \setminus a$. Then $a \cup b'$ and $a \cup b''$ are simple closed curves, at least one of which is essential. Call this essential curve β' .

If a approaches β from the same side at both endpoints, then β' is homotopic to, and disjoint from, a curve β'' which is disjoint from β . If instead a approaches β from opposite sides at both endpoints, one can instead find such a curve β'' that intersects β in a single point.

Thus, this curve β'' obtained from β by surgery along a has distance at most 2 from β in $\mathcal{C}^\dagger(S)$. By construction, β'' can be taken to intersect α in strictly fewer points than β , which shows Lemma 2.7 by induction. \square

Remark 2.8. The bound given in Lemma 2.7 is far from optimal. In fact, one could also bound the distance d^\dagger by the logarithm of the intersection number (as in the case of usual curve graphs, see [Hem01]). Namely, by choosing the right subarc a , one can ensure that both choices for β'' are essential, so we can halve the number of intersections in each step.

2.3. Rotation sets for torus homeomorphisms. Here and in what follows, we view the torus with its standard Euclidean structure $T = \mathbb{R}^2/\mathbb{Z}^2$.

Definition 2.9. Let $\tilde{f} \in \text{Homeo}_0(\mathbb{R}^2)$ be a lift of an isotopically trivial homeomorphism of T . The *rotation set* $\rho(\tilde{f}) \subset \mathbb{R}^2$ is the set of vectors

$$\rho(\tilde{f}) := \left\{ v \in \mathbb{R}^2 : \exists x_i \in \mathbb{R}^2, n_i \rightarrow \infty \text{ s.t. } \frac{(\tilde{f}^{n_i}(x_i) - x_i)}{n_i} \rightarrow v \right\}.$$

We recall some basic properties of ρ . The following are easy consequences of the definition:

- (1) For $p \in \mathbb{Z}^2$ and $n \in \mathbb{Z}$, we have

$$\rho(\tilde{f}^n + p) = n \rho(\tilde{f}) + p$$

- (2) For any lift \tilde{g} of an isotopically trivial torus homeomorphism, we have

$$\rho(\tilde{g}\tilde{f}\tilde{g}^{-1}) = \rho(\tilde{f}),$$

and

- (3) For any matrix $A \in SL_2(\mathbb{Z})$, we have

$$\rho(A\tilde{f}A^{-1}) = A(\rho(\tilde{f})).$$

Another important and much less trivial property is the result of Misiurewicz and Ziemian [MZ89] that $\rho(\tilde{f})$ is compact, convex, and equal to the convex hull of the *pointwise rotation set*

$$\rho_p(\tilde{f}) := \left\{ v \in \mathbb{R}^2 : \exists x \in \mathbb{R}^2, n_i \rightarrow \infty \text{ s.t. } \frac{(\tilde{f}^{n_i}(x) - x)}{n_i} \rightarrow v \right\}.$$

Thus, $\rho(\tilde{f})$ is either a point, a closed interval, or has non-empty interior. These, and many other properties of interest (for example, the property of containing a point with rational coordinates) are invariant under integer translations and therefore independent of the lift of f chosen. When considering such questions, we will often abuse notation and simply write $\rho(f)$, rather than $\rho(\tilde{f})$, thinking of $\rho(f)$ as a set well defined up to translation by \mathbb{Z}^2 .

As a byproduct of the proof of the convexity of the rotation set, Misiurewicz and Ziemian obtained the following result that we will need. Fix a fundamental domain $D \subset \mathbb{R}^2$. For any integer $n > 0$, let

$$\frac{1}{n}\tilde{f}^n(D) = \left\{ \frac{\tilde{f}^n(x)}{n} : x \in D \right\}.$$

Lemma 2.10 (Misiurewicz-Ziemian [MZ89]). *The sequence of compact subsets $(\frac{1}{n}\tilde{f}^n(D))$ converges to $\rho(\tilde{f})$ in the Hausdorff topology.*

As mentioned in the introduction, there is a rich and well-developed theory relating the dynamics of torus homeomorphisms to the geometric and topological properties of their rotation sets. A general introduction can be found in the original work of Misiurewicz and Ziemian [MZ89], and a more detailed description of recent developments in the survey [Beg07] (in French). We will need the following two important results.

Theorem 2.11 (Franks [Fra89]). *If $(a/q, b/q)$ is a point with rational coordinates in the interior of $\rho(\tilde{f})$, then f has a periodic point of period q .*

Theorem 2.12 (Llibre–MacKay, [LM91]). *Suppose $f \in \text{Homeo}_0(T)$ has rotation set with non-empty interior. Then there exists a finite f -invariant set P such that the restriction of f to $T - P$ is a pseudo-Anosov mapping class.*

The existence of the finite set P comes from Franks’s theorem quoted above. Llibre–MacKay’s insight was to show that the orbits of any three periodic points with non-collinear rational rotation vectors gives the desired set P with the pseudo-Anosov property.

Dehn twists. As discussed in [Doe97], homeomorphisms of T isotopic to a Dehn twist map also have a rotation set relative to the direction of the twist. We explain this now.

Definition 2.13. Let α be a curve of T . We say a homeomorphism f of T is a *Dehn twist map around α* when the following hold

- (1) the support of f is contained in an embedded annulus $A \subset T$ which contains α , and
- (2) there exists a chart $\varphi: A \rightarrow \mathbb{R}/\mathbb{Z} \times [0, 1]$ which sends α to $\mathbb{R}/\mathbb{Z} \times \{1/2\}$ and $r \in \mathbb{Z} \setminus \{0\}$ such that $\varphi f|_A \varphi^{-1} = \tau^r$, where

$$\begin{aligned} \tau: \quad \mathbb{R}/\mathbb{Z} \times [0, 1] &\rightarrow \mathbb{R}/\mathbb{Z} \times [0, 1] \\ (x, y) &\mapsto (x + y, y) \end{aligned} .$$

Fix a simple (oriented) essential loop α and let \check{T} denote the associated cyclic cover of T . We fix an identification of \check{T} with $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, with deck group equal to the integral translations in the second coordinate, oriented so that a lift of α is oriented in the positive direction. Let $\check{f} \in \text{Homeo}(\check{T})$ be isotopic to a Dehn twist map around α , and let $\check{f}: \check{T} \rightarrow \check{T}$ be a lift of f . Let $p_2: \check{T} = \mathbb{R}/\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ be the projection to the second co-ordinate, and

$$\begin{aligned} t: \quad \check{T} &\rightarrow \check{T} \\ (x, y) &\mapsto (x, y + 1) \end{aligned} .$$

Definition 2.14. [Doe97] The rotation set $\rho_\alpha(\check{f})$ of the lift \check{f} is the subset of \mathbb{R} consisting of accumulation points of

$$\left\{ \frac{p_2(\check{f}^n(\check{x})) - p_2(\check{x})}{n} : n \geq 1 \text{ and } \check{x} \in \check{T} \right\} .$$

This set is a segment of \mathbb{R} . Similarly to the case of homeomorphisms isotopic to the identity, it follows from the definition that $\rho_\alpha(f) = \rho_{g\alpha}(gf g^{-1})$ for any homeomorphism g of T , and for any integers p and q , we have

$$\rho(t^p \check{f}^q) = q\rho(\check{f}) + p.$$

Thus, two lifts of f to \check{T} have rotation sets which differ by an integral translation of \mathbb{R} , and so we use $\rho_\alpha(f)$ (and occasionally $\rho(f)$ when α is understood), rather than $\rho_\alpha(\check{f})$ to mean the rotation set of any fixed lift when we wish to speak of properties independent of choice of lift.

3. ASYMPTOTIC TRANSLATION LENGTH

In this section we prove continuity of asymptotic translation length. As a direct consequence, one may conclude that hyperbolicity for the action on $\mathcal{C}^\dagger(S)$ is an open condition on $\text{Homeo}(S)$. Before embarking on the proof, we take a brief detour to discuss the contrasting results that there are hyperbolic elements arbitrarily close to the identity, as well as open sets of elliptic elements.

Here, and for the remainder of the paper, $|\cdot|$ always denotes asymptotic translation length on the curve graph $\mathcal{C}^\dagger(S)$.

Lemma 3.1. *For any closed surface S of genus at least 1, any C^0 neighbourhood of the identity contains homeomorphisms acting hyperbolically on $\mathcal{C}^\dagger(S)$.*

Proof. First assume that S has genus at least 2. Recall that a *filling* loop γ means that every loop homotopic to γ intersects every essential simple closed curve on S . Let $p \in S$ be a point, and let $\gamma: [0, 1] \rightarrow S$ be a smooth, filling loop based at p , with transverse self-intersections.

Next, choose times $0 = t_0 < \dots < t_n = 1$ so that $\gamma|_{[t_i, t_{i+1}]}$ has length at most ϵ . By possibly adding more times before self-intersection points, we may assume that γ can be covered with disks D_1, \dots, D_{2k} so that

- (1) Each D_i has diameter $< \epsilon$,
- (2) $D_i \cap D_j = \emptyset$ if $i - j$ is even, and
- (3) $\gamma|_{[t_i, t_{i+1}]} \subset D_i$ for all i .

We let G_0 (respectively, G_1) be the endpoint of the isotopy supported in the union of all D_i for i even (respectively, odd) and in each such D_i slides $\gamma(t_i)$ to $\gamma(t_{i+1})$ along $\gamma|_{[t_i, t_{i+1}]}$. By choosing ϵ small enough,

$$F = G_1 \circ G_0$$

is arbitrarily close to the identity. We claim that F acts on $\mathcal{C}^\dagger(S)$ hyperbolically. To this end, it suffices to show that some power of F does. Observe that F^k is a homeomorphism of S fixing p , and (by composing the isotopies defining G_0, G_1), there is an isotopy from F^k to id whose trace of the point p is exactly the loop γ . This implies that the isotopy class $[F^k]$ lies in the kernel of the forgetful map

$$1 \rightarrow \pi_1(S, p) \rightarrow \text{Mcg}(S - p) \rightarrow \text{Mcg}(S) \rightarrow 1$$

and corresponds exactly to the loop $\gamma \in \pi_1(S, p)$ (compare [FM12, Section 4.2] for this Birman exact sequence). Since γ is filling, Kra's theorem [Kra81] implies that the mapping class $[F^k] \in \text{Mcg}(S - p)$ is pseudo-Anosov. Theorem 2.5 then implies that F^k (hence F) acts hyperbolically on $\mathcal{C}^\dagger(S)$.

If $S = T$ is the torus, we need to choose γ to be filling in $T \setminus \{q\}$ for some point q (as the Birman exact sequence requires at least one puncture in the case of the torus), and argue analogously. \square

Remark 3.2. We emphasise that the construction in Lemma 3.1 is flexible – the choice of the filling loop γ is arbitrary, and we can construct a root of the corresponding point-pushing pseudo-Anosov.

The following gives a general construction of open sets of elliptic elements which can be taken arbitrarily close to (though not containing) the identity.

Construction 3.3 (Open sets of elliptics). Let $A \subset S$ be an embedded essential, closed annulus, and $f : S \rightarrow S$ a homeomorphism isotopic to the identity and sending A into the interior of A . Let α denote a boundary curve of A . There exists a neighborhood of f in $\text{Homeo}_0(S)$ consisting of homeomorphisms which send A into the interior of A . Any such homeomorphism g will satisfy that $g^N(\alpha) \cap \alpha = \emptyset$, giving a bounded orbit.

3.1. Proof of Theorem 1.2. We now prove the following result.

Theorem 3.4. *Let $f_m \rightarrow f$ in the C^0 -topology on $\text{Homeo}(S)$. Then*

$$|f_m| \rightarrow |f|.$$

In particular, the set of homeomorphisms acting hyperbolically on $C^\dagger(S)$ is an open set in the C^0 -topology.

This is the main technical content of Theorem 1.2 stated in the introduction, indeed we have:

Proof of Theorem 1.2 given Theorem 3.4. Openness of hyperbolicity is immediate from the continuity of asymptotic translation length. What remains to show is that all positive real values are realized. In [BHW21], it is shown that taking powers of elements which act as pseudo-Anosov homeomorphisms relative to a fixed finite set produces elements with arbitrarily large translation length. Since $\text{Homeo}_0(S)$ is connected, this implies all positive real values are attained. \square

The proof of Theorem 3.4 will occupy the rest of the section. We begin with the following elementary lemma.

Lemma 3.5. *Suppose that g_m is a sequence of homeomorphisms converging to a homeomorphism g in the C^0 -topology, and let α be an essential simple closed curve. Then, for all m large enough, we have*

$$d^\dagger(g_m\alpha, g\alpha) \leq 2.$$

Proof. Let C be a small collar neighbourhood of $g\alpha$, so that the boundary of C consists of two simple closed curves disjoint from $g\alpha$. Then, for all m large enough, we have

$$g_m\alpha \subset \text{int}(C),$$

which implies the claim. \square

We now pick a sequence f_m with limit f as in Theorem 3.4.

Lemma 3.6. *Given any $n \in \mathbb{N}$ and essential simple closed curve α , for all sufficiently large m we have*

$$n|f_m| = |f_m^n| \leq d^\dagger(\alpha, f^n\alpha) + 2.$$

Proof. The first equality is immediate from the definition of asymptotic translation length. For the inequality, we have

$$|f_m^n| \leq d^\dagger(\alpha, f_m^n\alpha) \leq d^\dagger(\alpha, f^n\alpha) + d^\dagger(f^n\alpha, f_m^n\alpha),$$

and since $f_m^n \rightarrow f^n$ as $m \rightarrow \infty$, we invoke Lemma 3.5, which shows that $d^\dagger(f^n\alpha, f_m^n\alpha) \leq 2$ for sufficiently large m , and we are done. \square

As a consequence of this lemma we have

$$\limsup_{m \rightarrow \infty} |f_m| \leq |f|$$

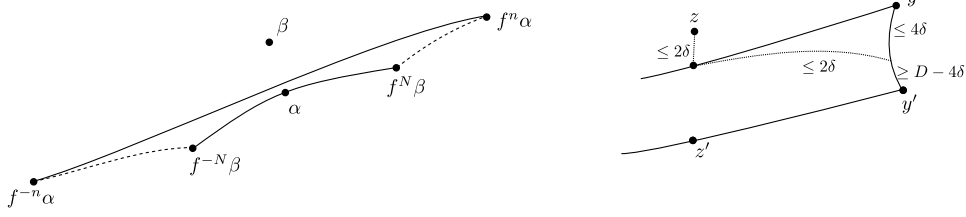


FIGURE 1. The situation in the proof of Lemma 3.7 (left) and 3.8 (right)

In particular, if $|f| = 0$, we have $\lim_{m \rightarrow \infty} |f_m| = 0 = |f|$ and we are done. Hence, we may from now on assume that $|f| > 0$ and we aim to prove that $\liminf_{m \rightarrow \infty} |f_m| \geq |f|$.

Let δ be a hyperbolicity constant for $\mathcal{C}^\dagger(S)$, so that any geodesic triangle in $\mathcal{C}^\dagger(S)$ is δ -slim, and any geodesic quadrilateral is 2δ -slim.

The first step will be to choose a convenient curve α to serve as a base-point.

Lemma 3.7. *There is $\alpha \in \mathcal{C}^\dagger(S)$ and $N \in \mathbb{N}$ such that for all $n \geq N$ we have*

$$d^\dagger(\alpha, [f^{-n}\alpha, f^n\alpha]) \leq 2\delta,$$

and furthermore, for sufficiently large m ,

$$d^\dagger(\alpha, [f_m^{-N}\alpha, f_m^N\alpha]) \leq 2\delta.$$

Proof. Start with any $\beta \in \mathcal{C}^\dagger(S)$. Then the sequence $(f^n\beta)_n$ is a C -quasi-geodesic for some $C > 0$. By [BH99, Theorem III.H.1.7.] (quasi-geodesics fellow travel geodesics) there exists a constant $B = B(\delta, C)$ such that any geodesic segment $[f^i\beta, f^j\beta]$ lies within Hausdorff distance B of $(f^k\beta)_{i \leq k \leq j}$.

Take any $N \in \mathbb{N}$ satisfying

$$N > \max \left\{ \frac{3B + 3\delta}{|f|}, \frac{2\delta + B + 4}{|f|} \right\}$$

and choose α to be a closest-point projection of β to $[f^{-N}\beta, f^N\beta]$. We have that $d^\dagger(\beta, \alpha) \leq B$. Fix $n > N$ and consider a geodesic segment $[f^{-n}\alpha, f^n\alpha]$ with $n \geq N$. Now we study the geodesic quadrilateral given by the aforementioned two geodesics together with $[f^{-N}\beta, f^{-n}\alpha]$ and $[f^N\beta, f^n\alpha]$, as indicated in the left side of Figure 1.

Since quadrilaterals are 2δ -slim, the point $\alpha \in [f^{-n}\alpha, f^n\alpha]$ has distance at most 2δ to one of the other three sides. To prove the lemma we want to show that α is not in the 2δ -neighborhood of $[f^{-N}\beta, f^{-n}\alpha]$ or of $[f^N\beta, f^n\alpha]$.

So, suppose for contraction that α is within 2δ of $[f^N\beta, f^n\alpha]$ (the other case is analogous). Since $d^\dagger(\alpha, \beta) \leq B$, we have

$$[f^N\beta, f^n\alpha] \subset N_{B+\delta}([f^N\beta, f^n\beta])$$

Therefore, $[f^N\beta, f^n\alpha]$ is in a $(2B + \delta)$ -neighborhood of $(f^i\beta)_{N \leq i \leq n}$. Since α is within 2δ of $[f^N\beta, f^n\alpha]$, this shows that β has distance at most $3B + 3\delta$ to

a point in $(f^i\beta)_{N \leq i \leq n}$. This contradicts the first lower bound in our choice of N , since $|i| \cdot |f| \leq d^\dagger(f^i\beta, \beta) \leq 3B + 3\delta$, but on the other hand we have $N \leq i$ by definition of i .

Now we tackle the last claim of the lemma, using the fact that

$$N > \frac{2\delta + B + 4}{|f|}.$$

Pick α as before. Then for sufficiently large m for $i \in \{-N, N\}$ we have that $d^\dagger(f_m^i\alpha, f^i\alpha) \leq 2$. We consider the geodesic quadrilateral with vertices $f_m^{-N}\alpha$, $f^{-N}\beta$, $f^N\beta$, and $f_m^N\alpha$. Recall that α lies on the geodesic $[f^{-N}\beta, f^N\beta]$. As before, if α is within 2δ of $[f^N\beta, f_m^N\alpha]$, then α is within $2\delta + B + 4$ of $f^N\alpha$, which is a contradiction. \square

Lemma 3.8. *If N is chosen large enough, and m is sufficiently large (depending on N), then for all $n \in \mathbb{N}$ and i with $-n \leq i \leq n$, we have that $(f_m^N)^i\alpha$ is within 4δ of $[(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]$.*

Proof. Here, we require in addition to the constraints of the previous proof that $N > \frac{8\delta+2}{|f|}$.

Given $n \in \mathbb{N}$, fix a geodesic segment $[(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]$ and let i be an index with $-n \leq i \leq n$ such that that $z := (f_m^N)^i\alpha$ maximises distance to $[(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]$. Let D denote the distance from z to $[(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]$.

Let $x := (f_m^N)^{i-1}\alpha$ and let $y := (f_m^N)^{i+1}\alpha$. Let z', x' and y' be choices of closest-point projections of z, x and y , respectively, to $[(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]$. Finally, let \bar{z} be a closest point projection of z to $[x, y]$, and recall that z is within 2δ of $[x, y]$ by Lemma 3.7.

Now consider the geodesic quadrilateral $[x, y], [y, y'], [y', x'], [x', x]$. By 2δ -slimness, $\bar{z} \in [x, y]$ is 2δ -close to one of the other three sides. To finish the proof, what we need to show is that the distance from \bar{z} to $[x', y']$ is at most 2δ . Similarly to the previous lemma, we proceed by using a proof by contradiction to show that \bar{z} cannot be close to one of the other two sides.

So, for contradiction, assume that \bar{z} is 2δ -close to a point $\bar{y} \in [y, y']$ (the case of $[x, x']$ is analogous). The set-up is illustrated in the right side of Figure 1.

We then have

$$D = d(z, [(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]) \leq 4\delta + d(\bar{y}, [(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]),$$

and therefore

$$d^\dagger(\bar{y}, [(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]) \geq D - 4\delta.$$

On the other hand, because our choice of z was a point that maximised distance to $[(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]$, we have that

$$d^\dagger(y, [(f_m^N)^{-n}\alpha, (f_m^N)^n\alpha]) \leq D,$$

hence

$$d^\dagger(\bar{y}, y) \leq 4\delta.$$

Thus we obtain in total that

$$d^\dagger((f_m^N)^i \alpha, (f_m^N)^{i+1} \alpha) = d(z, y) \leq 8\delta,$$

hence

$$d^\dagger(\alpha, f^N \alpha) \leq 8\delta + 2.$$

This contradicts $N > \frac{8\delta+2}{|f|}$, completing the proof. \square

Remark 3.9. Observe that the proof of Lemma 3.8 also shows that for any large enough N , the closest-point projections of the $(f_m^N)^i \alpha$ for $-n \leq i \leq n$ on the geodesic $[(f_m^N)^{-n} \alpha, (f_m^N)^n \alpha]$ are monotonic.

Lemma 3.10. *For all sufficiently large N , and sufficiently large m (depending on N), for all $n \in \mathbb{N}$ we have*

$$2n(|f^N| - 2 - 8\delta) \leq d^\dagger((f_m^N)^{-n} \alpha, (f_m^N)^n \alpha).$$

Therefore, $|f_m| \geq |f| - \frac{2+8\delta}{N}$.

Proof. Divide the segments of a geodesic $[(f_m^N)^{-n} \alpha, (f_m^N)^n \alpha]$ into $2n$ disjoint pieces by taking closest-point projections of the points $(f_m^N)^i \alpha$ to the geodesic (by the monotonicity of projections guaranteed by Remark 3.9). Each piece has length at least $|f^N| - 2 - 8\delta$ by Lemma 3.8 and Lemma 3.6 so we obtain the first inequality. The second inequality follows by rearranging the first inequality and taking the limit of this quantity over n as n tends to ∞ . \square

As a consequence of the above lemma, we obtain

$$\liminf_{m \rightarrow \infty} |f_m| \geq |f| - \frac{2 + 8\delta}{N}$$

for all sufficiently large N . This shows that $\liminf_{m \rightarrow \infty} |f_m| \geq |f|$, and Theorem 3.4 follows. \square

4. DISTANCE ESTIMATES

In this section, we provide key results which allow us to connect the geometry of $\mathcal{C}^\dagger(S)$ to the topology of curves on S and their lifts to specific covers.

Our first criterion, Lemma 4.1 below, works for all surfaces S of genus $g \geq 2$, and uses covering spaces. One can think of this as a $\mathcal{C}^\dagger(S)$ -version of the criterion introduced by Hempel in [Hem01, Section 2]. To state it, we need to introduce some vocabulary and basic observations. If $\pi: S' \rightarrow S$ is a (possibly branched) cover, and $\alpha \subset S$ is an essential simple closed curve (disjoint from the branch points), then we say that an *elevation* of α is a connected component of $\pi^{-1}(\alpha)$. The following two properties are obvious, yet important:

- (1) Elevations of simple closed curves are simple. If the cover has finite degree, an elevation of such a curve is also closed.
- (2) Any two elevations of disjoint simple closed curves are disjoint.

Lemma 4.1. *Let S be a surface of genus $g \geq 2$. If α and β are two curves with $d^\dagger(\alpha, \beta) = 2$, then there is a degree 2 cover $X \rightarrow S$ such that α and β admit disjoint elevations in X .*

Proof. Since $d^\dagger(\alpha, \beta) = 2$, there is a curve γ which is disjoint from both α and β . The following argument shows that we may take such a curve to be non-separating. Since $d^\dagger(\alpha, \beta) \neq 1$, the curves intersect, and so if γ were separating, then α, β would be contained in the same complementary component of γ . Thus, we could replace γ with any non-separating curve in the other complementary component.

Taking the mod-2 intersection number with γ gives a homomorphism $\pi_1(S) \rightarrow \mathbb{Z}/2\mathbb{Z}$; let X be the associated degree 2 cover. We claim that X has the desired property, and in fact α, β even admit disjoint lifts to X . Let γ_1 and γ_2 denote the two elevations of γ to X . Then

$$X - (\gamma_1 \cup \gamma_2) = X_1 \cup X_2$$

where X_1 and X_2 are the two preimages of $S - \gamma$. Now, the desired elevations can be obtained by lifting of α into X_1 , and β into X_2 . \square

Iterating Lemma 4.1 immediately yields the following.

Corollary 4.2. *If S is a surface of genus $g \geq 2$ and $\alpha, \beta \in \mathcal{C}^\dagger(S)$ with $d^\dagger(\alpha, \beta) \leq n$, then there is a cover of S of degree at most 2^n to which α, β admit disjoint elevations.*

As a consequence, we have the following criterion that we will frequently apply later.

Lemma 4.3. *Let S be a surface of genus $g \geq 2$ and let $K \geq 0$. There is a finite-sheeted cover $X \rightarrow S$ (depending only on K) such that any two curves α and β on S satisfying $d^\dagger(\alpha, \beta) \leq K$ admit disjoint elevations to X .*

Proof. Let $\Gamma < \pi_1(S)$ be the intersection of the finite index subgroups of degree at most 2^K in $\pi_1(S)$. It is a well-known fact that a finitely generated group has only finitely many subgroups a given index (e.g. since an index n subgroup H of G is determined by the action of G on the cosets of H , and there are only finitely many homomorphisms of a finitely generated group to the symmetric group of n elements). Thus, Γ also has finite index in $\pi_1(S)$. By covering space theory, the subgroup Γ determines a finite cover $X \rightarrow S$ of degree equal to the index of Γ . We claim this cover has the desired property. Indeed, by Corollary 4.2, any pair of curves α and β satisfying $d^\dagger(\alpha, \beta) \leq K$ will admit disjoint elevations to some cover $X' \rightarrow S$ where $X' \rightarrow S$ has degree at most 2^K . By definition of Γ , we have $\Gamma < \pi_1(X') < \pi_1(S)$, and hence $X \rightarrow S$ factors through $X \rightarrow X' \rightarrow S$. Disjoint elevations stay disjoint in further covers, hence α and β have disjoint elevations to X as required. \square

4.1. A crossing number estimate. We also develop an upper bound on distance in $\mathcal{C}^\dagger(T)$ (Lemma 4.5, below) specifically for the torus. This will be used in our work on rotation sets in the next section.

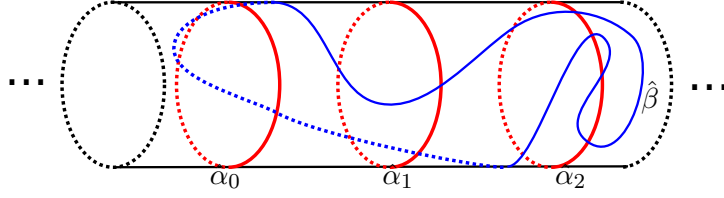


FIGURE 2. Crossing number via annuli

Definition 4.4. Let α and β be essential simple closed curves on the torus $T = S^1 \times S^1$. Interpret β as a map $\beta: [0, 1] \rightarrow T$ and denote by $\tilde{\beta}: [0, 1] \rightarrow \tilde{T} = \mathbb{R}^2$ a lift.

We define the α -crossing number $C_\alpha(\beta)$ as the number of distinct elevations of α which $\tilde{\beta}$ intersects.

When α and β are homotopic essential simple closed curves, there is a useful alternative way to describe $C_\alpha(\beta)$. Namely, let A be the annular covering of T corresponding to the cyclic subgroup generated by α . We can identify A with $S^1 \times \mathbb{R}$ so that the lifts $\hat{\alpha}_m$ of α are the circles $S^1 \times \{m\}$, $m \in \mathbb{Z}$.

Then the crossing number $C_\alpha\beta$ is equal to the cardinality of

$$\{\hat{\alpha}_m : \hat{\beta} \cap \hat{\alpha}_m \neq \emptyset\},$$

where $\hat{\beta}$ is an elevation of β . In other words, the number of lifts of α which $\hat{\beta}$ intersects (compare Figure 2). Observe that this last characterisation is independent of the identification of the annulus with $S^1 \times \mathbb{R}$.

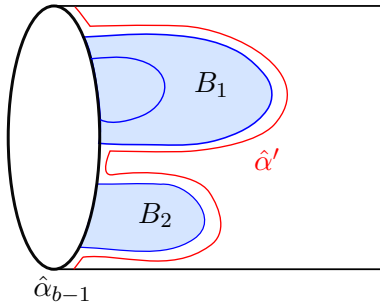
Lemma 4.5. *If α and β are isotopic simple closed curves on T , then*

$$C_\alpha(\beta) + 1 \geq d^\dagger(\alpha, \beta).$$

Proof. We will perform a surgery replacing α with a homotopic curve α' such that $i(\alpha, \alpha') = 0$ and $C_{\alpha'}(\beta) \leq C_\alpha(\beta) - 1$. By induction, this is enough to prove the lemma.

To this end, we fix the cover A and a lift $\hat{\beta}$ as in the discussion before the lemma, and keep the notation $\hat{\alpha}_m = S^1 \times \{m\}$ for the lifts of α . Choose $a < b \in \mathbb{Z}$ with $b - a$ minimal, so that $\hat{\beta} \subset S^1 \times [a, b]$. Note that, if $\hat{\beta}$ intersects $\hat{\alpha}_b = S^1 \times \{b\}$, then we may replace α with a nearby parallel copy of itself α' avoiding these finitely many points or intervals of intersection, and decreasing crossing number, thus already satisfying our desired outcome. Thus, we may assume $\hat{\beta}$ does not intersect $S^1 \times \{b\}$. Consider the intersection of $\hat{\beta}$ with $S^1 \times [b - 1, \infty)$. This is a collection of arcs b_1, \dots, b_k with each arc intersecting $\hat{\alpha}_{b-1}$ only in its endpoints. Choose disjoint, closed disks B_1, \dots, B_r with the following properties:

- (1) Each B_i is bounded by a segment contained in $\hat{\alpha}_{b-1}$, and one of the b_j .
- (2) Each b_j is contained in one of the B_i .


 FIGURE 3. Surgery of α along arcs of β

- (3) All B_i are disjoint from $\hat{\alpha}_i, i \neq b - 1$.

By the third property, the B_i map to disjoint, embedded disks in T under the covering map.

We let α' be the curve obtained from surgery of α at all b_j which appear as boundary segments of B_i . Since this surgery can be done on A , α' is homotopic to α . Since all b_j approach $\hat{\alpha}_{b-1}$ from the same side, the surgered curve α' can be chosen to be disjoint from α . Hence, the cover defined by α' is still A . The lifts $\hat{\alpha}'_i$ are obtained from $\hat{\alpha}_i$ by surgering at deck group translates of the B_i .

Thus, the lift $\hat{\beta}$ can only intersect $\hat{\alpha}'_i$ if it also intersects $\hat{\alpha}_i$. By property 2, $\hat{\beta}$ does not intersect $\hat{\alpha}'_{b-1}$. Thus, we have $C_{\alpha'}(\beta) \leq C_{\alpha}(\beta) - 1$ by the description of crossing number before the lemma. \square

Remark 4.6. One could define a version of crossing number also for surfaces of higher genus by using the Bass–Serre tree of the cyclic splitting of the fundamental group determined by α . In this case the diameter of the projection to the Bass–Serre tree will provide an upper bound on distance, the idea of the proof is the same. However, since we do not need this for our intended applications, we do not pursue this here.

5. HYPERBOLIC ELEMENTS

In this section, we will study homeomorphisms isotopic to the identity acting hyperbolically on \mathcal{C}^\dagger and prove Theorem 1.3 from the introduction.

5.1. Proof of Theorem 1.3. Recall that Theorem 1.3 asserts the equivalence of the following statements for $f \in \text{Homeo}_0(T)$:

- (1) f acts hyperbolically on $\mathcal{C}^\dagger(T)$,
- (2) the rotation set $\rho(f)$ of f has non-empty interior, and
- (3) there is a finite f -invariant set $P \subset T$ such that f represents a pseudo-Anosov mapping class of $T - P$.

For the proof, we will need the following consequence of Lemma 4.5.

Lemma 5.1. *If f acts hyperbolically on $\mathcal{C}^\dagger(T)$ then for any curve $\alpha \in \mathcal{C}^\dagger(T)$ and for all $n > 0$ we have*

$$C_\alpha(f^n(\alpha)) \geq |f| \cdot n - 1.$$

Proof. Using Lemma 4.5 we have for all $n > 0$

$$|f| \cdot n \leq d^\dagger(\alpha, f^n(\alpha)) \leq C_\alpha(f^n(\alpha)) + 1. \quad \square$$

Proof of Theorem 1.3. The assertion (2) \Rightarrow (3) is Theorem 2.12 of Llibre–MacKay [LM91], and (3) \Rightarrow (1) is Theorem 2.5. Thus, we need only show the implication (1) \Rightarrow (2).

Suppose f acts hyperbolically on $\mathcal{C}^\dagger(T)$. Identify T with $\mathbb{R}^2/\mathbb{Z}^2$, and let α be the simple closed curve whose lifts to \mathbb{R}^2 are the horizontal lines in \mathbb{R}^2 with integer second co-ordinates. Let \tilde{f} be a lift of f to \mathbb{R}^2 .

By Lemma 5.1 we have

$$C_\alpha(f^n(\alpha)) \geq |f| \cdot n - 1.$$

This means that for any n , lifts of $f^n(\alpha)$ intersect at least $|f| \cdot n - 1$ distinct horizontal lines (with integer second co-ordinate) in the plane \mathbb{R}^2 . In other words, there are sequences of points z_n and z'_n (on the same horizontal line) so that, for each $n \in \mathbb{N}$ the points $\tilde{f}^n(z_n)$ and $\tilde{f}^n(z'_n)$ have y -coordinate differing by at least $|f| \cdot n - 1$, hence the displacements $d_n := \tilde{f}^n(z_n) - z_n$ and $d'_n := \tilde{f}^n(z'_n) - z'_n$ have y -coordinates differing by at least $|f| \cdot n - 1$. This implies that the projection of $\rho(f)$ to the y -axis has diameter at least $|f|$.

Since rotation sets are convex and compact, we need only now rule out the possibility that $\rho(f)$ is a line segment. Suppose for contradiction that $\rho(f)$ were a line segment, ℓ . If ℓ has rational slope, then we may find $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $A(\ell)$ is a subset of the x -axis. As discussed in Section 2.3, we have $\rho(AfA^{-1}) = A\rho(f)$. Since asymptotic translation length is a conjugacy invariant, we have $|AfA^{-1}| = |f|$, so we may apply the same argument as above to conclude that projection of $\rho(AfA^{-1}) = A\rho(f)$ to the y -axis has diameter at least $|f|$, a contradiction. If the slope of ℓ is irrational, we can again find $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $A(\ell)$ has projection to the y -axis a set of diameter less than $|f|$: in fact, with a process similar to Euclid's algorithm, one may apply elementary matrices to make the projection of the length of the segment ℓ arbitrarily small, again contradicting the lower bound. This concludes the proof. \square

In the following proposition we bound the area of the rotation set from below by a constant multiple of the square of the asymptotic translation length on $\mathcal{C}^\dagger(S)$. There is no analogous upper bound because there are rotation sets with arbitrarily large area but of bounded height, and height bounds the translation length from above by Lemma 5.1.

Proposition 5.2. *Let $f \in \text{Homeo}_0(T)$ act hyperbolically on $\mathcal{C}^\dagger(T)$. Then*

$$\text{Area}\rho(f) \geq \frac{\sqrt{3}}{8}|f|^2.$$

Proof. First note that $\rho(f)$ is compact, and has non-empty interior by Theorem 1.3, so we may pick a maximal area parallelogram $P \subset \rho(f)$. Now find $A \in \text{SL}_2(\mathbb{R})$ such that the diagonals of the parallelogram $A \cdot P$ are horizontal/vertical, and of the same length. Thus $A \cdot P$ is equal to a closed r -ball in the L^1 norm centered at the intersection of the diagonals Q . Since $A \cdot \rho(f)$ is also convex, and $A \cdot P \subset A \cdot \rho(f)$ has maximal area, we must have that $A \cdot \rho(f)$ is contained within the closed r -ball in the L^∞ norm with the same centre Q . This in turn is contained in a disk D of radius $\sqrt{2}r$. Thus $\text{Area}D = \pi \text{Area}P$. It suffices to bound the diameter of D from below to establish a lower bound on $\text{Area}\rho(f)$.

After post composing by a rotation, we may assume that $(h, 0) \in A \cdot \mathbb{Z}^2$ is a non-zero vector of smallest length and $h > 0$. It is well known that $h^2 \leq \frac{2}{\sqrt{3}}$. Let $(k, v) \in A \cdot \mathbb{Z}^2$ be a smallest length vector outside the span of $(h, 0)$. Then we must have $v = \frac{1}{h}$ because A is area preserving. Now D contains $A \cdot \rho(f)$, so by Lemma 5.1 the projection of D to the second co-ordinate is at least $\frac{1}{h} \cdot |f|$. This gives the desired lower bound on $\text{Area}D = \pi \text{Area}P$ and hence $\text{Area}\rho(f)$ as required. \square

5.2. General characterisation of hyperbolic isometries. In this section we prove the following extensions of Theorem 1.3.

Theorem 5.3 (Characterisation of hyperbolic homeomorphisms). *Let f be an orientation preserving homeomorphism of T . Then f acts hyperbolically on $\mathcal{C}^\dagger(T)$ if and only if f satisfies one of the following (mutually exclusive) conditions.*

- (1) *The homeomorphism f is isotopic to an Anosov homeomorphism of T ,*
- (2) *a finite power of f is isotopic to a Dehn twist map about some simple closed curve α , and its rotation set ρ_α has non-empty interior, or*
- (3) *a finite power of f is isotopic to the identity and its rotation set has non-empty interior.*

Of course, if a homeomorphism f of T reverses the orientation, f acts hyperbolically on $\mathcal{C}^\dagger(T)$ if and only if the orientation-preserving homeomorphism f^2 does so. We hence have a complete characterisation of elements of $\text{Homeo}(T)$ which act hyperbolically on $\mathcal{C}^\dagger(T)$.

Proof. For the first statement, it is a special (and easy) case of the Nielsen–Thurston classification theorem that any orientation-preserving homeomorphism of T either is isotopic to an Anosov homeomorphism of T , or has a power isotopic to a Dehn twist map of T , or has a power isotopic to the identity. In the first case, it acts hyperbolically on $\mathcal{C}^\dagger(T)$ since it acts hyperbolically on the Farey graph $\mathcal{C}(T)$, and there is a 1-coarsely Lipschitz

map $\mathcal{C}^\dagger(T) \rightarrow \mathcal{C}(T)$ via sending α to its isotopy class $[\alpha]$. In the third case, Theorem 1.3 states that f acts hyperbolically if and only if its rotation set has nonempty interior. Thus, we need only understand the case where f is isotopic to a Dehn twist.

To this end, suppose f is isotopic to a Dehn twist around the essential simple closed curve α . Assume first that $\rho_\alpha(f)$ has non-empty interior. Then, by a theorem by Doeff (see [Doe97, Theorem 6.5]), f is isotopic to a pseudo-Anosov homeomorphism relative to a finite subset P of T . By Theorem 2.5, this implies that f acts hyperbolically on $\mathcal{C}^\dagger(T)$.

Now, suppose that $\rho_\alpha(f)$ has empty interior, in which case it is a singleton. With the same methods as for homeomorphisms isotopic to the identity, we will prove that f cannot act hyperbolically on $\mathcal{C}^\dagger(T)$. Denote by \check{T} the cyclic covering associated to α . Take any lift $\check{\alpha}$ of the loop α to \check{T} and fix a lift $\check{f}: \check{T} \rightarrow \check{T}$ of f . We identify \check{T} with $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, where the group of deck transformations of the covering map $\check{T} \rightarrow T$ is the group of integral translations of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$. Let $p_2: \check{T} = \mathbb{R}/\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ be the projection. For any $n \geq 0$, denote by D_n the diameter of $p_2(\check{f}^n(\check{\alpha}))$ and observe that $\lfloor D_n \rfloor \leq C_\alpha(f^n(\alpha)) \leq \lfloor D_n \rfloor + 1$.

As the rotation set of f is a singleton,

$$\lim_{n \rightarrow +\infty} \frac{D_n}{n} = 0$$

so

$$\lim_{n \rightarrow +\infty} \frac{C_\alpha(f^n(\alpha))}{n} = 0.$$

Hence, by Lemma 4.5,

$$\lim_{n \rightarrow +\infty} \frac{d^\dagger(\alpha, f^n(\alpha))}{n} = 0$$

and f does not act hyperbolically, concluding the proof of Theorem 5.3. \square

6. PARABOLIC AND ELLIPTIC ELEMENTS

Given Theorem 1.3, one might naïvely hope for a similar characterisation of elliptic and parabolic isometries of $\mathcal{C}^\dagger(T)$ in terms of their rotation sets. In this section we show this naïve hope is too optimistic, giving sufficient dynamical criteria for elliptic and parabolic actions on the torus, then showing why these cannot be necessary. Finally, we return from the torus to surfaces of higher genus and complete the proof of Theorem 1.1.

6.1. Sufficient dynamical criteria. Recall Theorem 1.4 is the claim that, for $f \in \text{Homeo}_0(T)$,

- (1) if $\rho(f)$ is an irrational slope segment, then f acts on $\mathcal{C}^\dagger(T)$ as a parabolic isometry, and
- (2) if $\rho(f)$ is a rational slope segment through a rational point, then the action of f is elliptic.

Proof of Theorem 1.4, rational slope through rational point case. This case follows from work of Dávalos [Dav18] and our crossing number estimates. Suppose that f is isotopic to the identity and the rotation set $\rho(f)$ is a rational slope segment through a rational point. By Theorem A in [Dav18], f is *annular*, which means that there is some $v \in \mathbb{Z}^2$ so that for any lift F of f to \mathbb{R}^2 we have

$$|\langle F^n(x) - x, v \rangle| \leq M$$

for some M and all $x \in \mathbb{R}^2$. This implies that the crossing numbers $C_\alpha(f^n\alpha)$ (as in Section 4) are bounded for suitable α , and thus f acts elliptically, as desired. \square

Remark 6.1. A similar argument can be applied to homeomorphisms isotopic to a Dehn twist map around an essential simple closed curve α whose rotation set consists of a single rational point. Denote by \tilde{T} the cyclic cover associated to α , by $\tilde{f}: \tilde{T} \rightarrow \tilde{T}$ a lift of f to this cyclic cover and, by t a positive generator of the group of deck transformations of $\tilde{T} \rightarrow T$. Then, if the rotation set of \tilde{f} is $\left\{\frac{p}{q}\right\}$, by a theorem by Addas-Zanata, Garcia and Tal (see [AZTG14, Theorem 2]), the homeomorphism $t^{-p}\tilde{f}^q$ has a compact invariant set which separates \tilde{T} . Hence f acts elliptically on $\mathcal{C}^\dagger(T)$.

To treat the case of irrational slope, we use the following construction.

Construction 6.2. For $n \in \mathbb{N}$, let $T_n \rightarrow T$ be the cover of the torus defined by reducing mod n in homology:

$$\pi_1(T) \rightarrow (\mathbb{Z}/n\mathbb{Z})^2$$

We equip T with the usual flat metric inherited from $T = \mathbb{R}^2/\mathbb{Z}^2$, and we equip T_n with the metric obtained by pulling back the metric of T to T_n , and rescaling by $1/n$. Observe that T and T_n are isometric, but not via the covering map.

The following lemma says that this covering behaves nicely with respect to the associated curve graphs:

Lemma 6.3. *Suppose that α and β are adjacent in $\mathcal{C}^\dagger(T)$. Then any choice of lifts of α and β in T_n are adjacent in $\mathcal{C}^\dagger(T_n)$.*

Proof. Let α' and β' be any choice of lifts in T_n of α and β . If α and β were disjoint then it is immediate that α' and β' are disjoint. On the other hand if $|\alpha \cap \beta| = 1$ then, after applying a homeomorphism of T (which lifts to the characteristic cover T_n), we can suppose that α and β are the standard generators of the fundamental group of $T = \mathbb{R}^2/\mathbb{Z}^2$. Then it is immediate that lifts α and β to T_n have only one intersection point and are thus adjacent in $\mathcal{C}^\dagger(T_n)$. \square

We also need the following consequence of Lemma 4.3.

Lemma 6.4. *Given any $K \geq 0$ there exists a translation surface $X = X(K)$ with the following property:*

For any n and any $p \in T_n$, there is a cover $f: X \rightarrow T_n$ branched only over p so that:

(1) *If α and β are curves on T_n disjoint from p with*

$$d^\dagger(\alpha, \beta) \leq K + 1,$$

then α, β admit disjoint elevations in the branched cover f .

(2) *The branched cover f is a local isometry at each nonsingular point.*

We emphasise that in the lemma, the *covering map* f may depend on p , but the geometry of the translation surface X does not. In addition, the geometry of X does not depend on n .

Proof. First, we show that given K and $p \in T_n$, there is a translation surface $X(K, p)$ which has the desired properties for T_n , we then show independence from n and p .

To prove this, begin by observing that there is a 3-fold branched cover $f_1: Y \rightarrow T_n$, branched only at p , where Y has genus 2. Suppose that $d^\dagger(\alpha, \beta) \leq K + 1$, and let α' and β' be any choice of elevations of α and β to Y . Observe that $d^\dagger(\alpha', \beta') \leq 2K + 2$ in $\mathcal{C}^\dagger(Y)$. This is because any choice of elevations of adjacent curves in T_n to Y will intersect at most three times, and therefore have distance at most 2 in $\mathcal{C}^\dagger(Y)$. Now we may apply Lemma 4.3 to find a finite-sheeted cover $f_2: X \rightarrow Y$, which only depends on K (and Y), such that any α' and β' as above admit disjoint elevations to X . The branched cover $f_p = f_1 \circ f_2: X(K, p) \rightarrow T_n$ then has (1). By pulling back the translation surface structure from T_n to $X(K, p)$, we can also satisfy (2).

Now if p' is any other point, observe that there is an isometry $\iota: T_n \rightarrow T_n$ mapping p to p' . Then $f_{p'} = \iota \circ f_1 \circ f_2: X(K, p) \rightarrow T_n$ has the desired property (1) for curves disjoint from the point p' . Since ι is an isometry, $f_{p'}$ also satisfies (2). Hence, the translation surface $X(K, p)$ can be chosen not to depend on p . Since all T_n are isometric, the surface $X(K)$ can also be chosen to be the same for all T_n . \square

Finally, we use the following observation, likely well known to experts. We include a short proof for completeness.

Lemma 6.5. *Let X be a compact square-tiled translation surface and $\lambda \notin \mathbb{Q}$. Then there exists $L > 0$ such that any straight line of length L and slope λ that is disjoint from the singularities of X will intersect all horizontal curves on X .*

Proof. Since λ is irrational, every half-leaf of the foliation with slope λ is dense in X . Supposing for contradiction that we can find a sequence of segments L_n of length $\ell_n \rightarrow \infty$ so that each L_n misses some horizontal curve c_n . We parametrize the segments L_n by arclength so that we see the

segments L_n as continuous maps $[-\ell_n/2, \ell_n/2] \rightarrow X$. After possibly taking a subsequence, we can assume that these L_n converge uniformly on any segment of \mathbb{R} to an infinite leaf μ of slope λ .

Since μ is dense and of constant slope, some compact subinterval $\mu' \subset \mu$ will intersect all horizontal curves, and we may find such a μ' which does not contain any singularity of X . Thus, by compactness, there exists $\epsilon > 0$ such that the ϵ -neighborhood N of μ' is disjoint from the singularities of X . Since the L_n converge to μ locally on compact subintervals, the intersection $L_n \cap N$ is non-empty for sufficiently large n . Now any connected component of $L_n \cap N$ which does not contain the endpoints of L_n is also a line of slope λ , and must intersect all horizontal curves because μ' can be isotoped along horizontal lines to a subset of $L_n \cap N$. But this contradicts the definition of L_n , completing the proof. \square

Proof of Theorem 1.4, irrational slope case. Suppose $\rho(f)$ is a line segment of irrational slope λ and length $\ell > 0$. By Theorem 1.3, the homeomorphism f is not hyperbolic, as the rotation set has empty interior. Thus, to prove the theorem, it suffices to show f is not elliptic. Let α be the standard horizontal curve on T . Supposing for contradiction that f is elliptic, there exists some K such that $d^\dagger(\alpha, f^n \alpha) \leq K$ for every n .

Consider the finite-sheeted covers $T_n \rightarrow T$ and metrics as described above. Let $D = [0, 1] \times [0, 1]$ denote the standard fundamental domain for $T = \mathbb{R}^2/\mathbb{Z}^2$. We also view T_n as the standard quotient $\mathbb{R}^2/\mathbb{Z}^2$ because T_n is isometric to T . We choose D_n to be the subset $[0, 1/n] \times [0, 1/n]$ in \mathbb{R}^2 . Fix a lift $\tilde{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of $f: T \rightarrow T$. Now we may also construct $\tilde{f}_n: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\tilde{f}_n(x) = \frac{1}{n} \tilde{f}(nx),$$

which descends to a map $\hat{f}_n: T_n \rightarrow T_n$, which is a lift of f .

Let $L > 0$ be the length guaranteed by Lemma 6.5 applied to the surface $X = X(K)$ guaranteed by Lemma 6.4 and slope λ , i.e. any straight line on X of slope λ and length L must intersect every horizontal curve on X .

Now fix some $k \in \mathbb{N}$ such that $k\ell > L$. By Lemma 2.10 we have that $\frac{1}{n} \tilde{f}^n(D)$ converges to $\rho(f)$ in the Hausdorff topology as $n \rightarrow \infty$. By definition we have that $\tilde{f}_n^{kn}(D_n) = \frac{1}{n} \tilde{f}^{kn}(D)$, and hence $\tilde{f}_n^{kn}(D_n)$ also converges to $k\rho(f)$. Let $\overline{k\rho_n} \subset T_n$ denote the projection of $k\rho(f)$, where we drop f from the notation for convenience.

We claim now that for each n there is a straight line a_n in T (disjoint from the standard horizontal curve α), with lift \hat{a}_n in T_n , such that the Hausdorff distance between $\hat{f}_n^{kn}(\hat{a}_n)$ and $\overline{k\rho_n}$ tends to 0 as n tends to ∞ . To find such a straight line, pick two points q_n and q'_n in $\tilde{f}_n^{kn}(D_n)$ which are close to the two ends of $\overline{k\rho_n}$. Then take as \hat{a}_n the straight line of D_n which joins the points $\hat{f}_n^{-kn}(q_n)$ and $\hat{f}_n^{-kn}(q'_n)$, and a_n its projection to T .

Now define simple closed curves α_n on T such that α_n contains a_n but is also isotopic to and disjoint from the standard horizontal curve α . For

large enough n we have that the Hausdorff distance between $\hat{f}_n^{kn}(D_n)$ and $\overline{k\rho_n}$ on T_n tends to 0 as n tends to ∞ . Write $\hat{\alpha}_n$ for the lift of α_n to T_n that contains \hat{a}_n , and let $\hat{\alpha}$ be an arbitrary lift of α to T_n .

By our initial assumption on f we have $d^\dagger(\alpha, f^{kn}\alpha_n) \leq K + 1$ for every $n > 0$. By Lemma 6.3 we also have $d^\dagger(\hat{\alpha}, \hat{f}_n^{kn}\hat{\alpha}_n) \leq K + 1$ for all $n > 0$ in $\mathcal{C}^\dagger(T_n)$. For any point p disjoint from both $\hat{\alpha}$ and $\hat{f}_n^{kn}\hat{\alpha}_n$, applying Lemma 6.4, we find that there are branched covers $\pi_n: X \rightarrow T_n$, so that $\hat{\alpha}$ and $\hat{f}_n^{kn}\hat{\alpha}_n$ admit disjoint lifts to X .

By our choices, there is a lift of $\hat{f}_n^{kn}\hat{a}_n$ to X which is disjoint from some horizontal curve in X for each n . But $\hat{f}_n^{kn}\hat{a}_n$ converges to $k\rho(f)$, which has slope λ and length $k\ell > L$. By choosing the branch point p outside a small embedded regular neighbourhood of $k\rho(f)$, we can guarantee that lifts of $\hat{f}_n^{kn}\hat{a}_n$ to X contain segments which converge to a segment with slope λ and length $k\ell > L$ as well. For sufficiently large n this contradicts Lemma 6.5. \square

6.2. Parabolics with a singleton rotation set. It is easy to produce examples of elliptic isometries with a given singleton rotation vector. Given (a, b) , the translation $(x, y) \mapsto (x + a, y + b)$ is a lift of a torus map with rotation set $\{(a, b)\}$. There are many dynamically more interesting examples as well. For instance, Koropecki and Tal constructed a homeomorphism of the torus whose rotation set is reduced to $\{(0, 0)\}$ but which has unbounded orbits in every direction in the universal cover (see [KT14]). Similar examples can be constructed with any rational one-point rotation set. We believe that their example acts as a parabolic isometry of $\mathcal{C}^\dagger(S)$.

In this section we give a construction that produces parabolic isometries with singleton rotation sets.

Proposition 6.6. *There are homeomorphisms f that act parabolically on $\mathcal{C}^\dagger(T)$ with $\rho(f) = \{(0, 0)\}$.*

As we remark later, a small modification of the construction can be used to produce singleton rotation sets other than $\{(0, 0)\}$. We first give the construction, then prove it is parabolic.

Construction 6.7. Start with a standard Denjoy counterexample map $D: S^1 \rightarrow S^1$ with irrational rotation number α , produced by blowing up a single orbit of a standard irrational rotation. See [Ghy01] for an introduction to these examples. We interpret $T = S^1 \times S^1$ as the mapping torus of D , and let φ_t be the suspension flow. The time-1-map φ_1 of this flow preserves the foliation of T into circles, acting with rotation number α on each of them. We think of the circles as being *horizontal*, and the flow lines as being *vertical*.

Let K be the suspension of the minimal invariant Cantor set of D ; the set K is invariant under the flow φ_t . Denote by λ the suspension of the boundary points. Now let J denote a closed interval in a horizontal circle

whose interior $\overset{\circ}{J}$ is a wandering interval for the Denjoy circle map. Then we obtain an embedding $\Phi: U = \overset{\circ}{J} \times \mathbb{R} \rightarrow T$ given by flowing this transverse segment, whose complement is exactly the suspension of the invariant Cantor set of D . We denote the coordinates given by this map by (x, t) . We now define a map preserving flow lines of U via the map

$$(x, t) \mapsto (x, t + \eta(x)e^{-|t|})$$

where $\eta(x) < 1$ is a bump function supported on J that is strictly positive on the interior. This map extends to continuously over K as the identity. We call the resulting map f . We will prove that it is the desired counterexample. Note that f preserves flow lines of φ_t , moving along lines monotonically and with exponentially decreasing speed as one approaches the set K .

Proposition 6.6 will be a consequence of the following two claims:

Claim 6.8. *The rotation set of f is $\{(0, 0)\}$.*

Claim 6.9. *The homeomorphism f acts parabolically on $\mathcal{C}^\dagger(T)$.*

Proof of Claim 6.8. Denote by $\tilde{\varphi}_t, \tilde{f}$ respective lifts of φ_t, f to the universal cover \mathbb{R}^2 of T . For any $p = (\tilde{x}, t) \in \mathbb{R}^2$ in (a lift of) U we have using growth condition (*) above that

$$\lim_{i \rightarrow \infty} \frac{\tilde{f}^{n_i}(p) - p}{n_i} < \lim_{i \rightarrow \infty} \frac{\tilde{f}^{n_i}(p_k) - p}{n_i} = \lim_{i \rightarrow \infty} \frac{\tilde{f}^{n_i}(p_k) - p_k}{n_i} < e^{-t_k} \rightarrow 0,$$

where $p_k = \tilde{f}^k(p) = (\tilde{x}, t_k)$ denotes an iterate of p so that $t_k \rightarrow \infty$ and the claim follows. \square

Proof of Claim 6.9. Given Claim 6.8 and Theorem 1.3, it follows that the homeomorphism is not hyperbolic.

We thus have to exclude the case that it is elliptic. To this end, suppose that it were. Let γ be a horizontal curve on T containing the segment J from property (4). By hypothesis, the sequence $(d^\dagger(\gamma, f^n(\gamma)))_n$ is bounded.

By Lemma 6.4, there exists a translation surface X and a finite cover $X \rightarrow T$ of bounded order, branched at a single point p (independent of n), so that, for any n , the curves $\gamma, f^n(\gamma)$ admit disjoint lifts $\tilde{\gamma}$ and $\tilde{\beta}_n$ to X and the cover is locally isometric. Note that we are free to pick p outside γ and $J \times \mathbb{R}$.

By Lemma 6.5, there is a number $L > 0$ so that any segment of slope α and length greater than L of X intersects every horizontal curve.

Claim 6.9 will be a consequence of the following technical claim.

Claim 6.10. *There exists a sequence $k_n \rightarrow +\infty$ of integers and a sequence (L_n) of straight segments of T of slope α such that*

- (1) *for any $n \geq 0$, there exists a subsegment σ_n of $f^{k_n}(J \times \{0\})$ such that $d_{\text{Hausdorff}}(L_n, \sigma_n) \rightarrow 0$.*
- (2) *the length of L_n tends to $+\infty$.*

(3) for any $n \geq 0$, the segment L_n does not meet the singularity p .

Before proving this claim, let us use it to prove Claim 6.9. For any $n \geq 0$, let $\tilde{\sigma}_n$ be the lift of σ_n to X which is contained in $\tilde{\beta}_n$, \tilde{L}_n be the lift of L_n which is close to $\tilde{\sigma}_n$. Taking a subsequence if necessary, the sequence \tilde{L}_n converges (on compact subsegments) to a straight line \tilde{L}_∞ of infinite length (like in the proof of Lemma 6.5). Take a subsegment L'_∞ of L_∞ which does not meet the singularities of the covering map and whose length is greater than L . Take $\epsilon > 0$ such that the ϵ -neighbourhood of L'_∞ does not meet the singularities of the covering map and such that any connected set at Hausdorff distance at most ϵ from L'_∞ meets any horizontal curve.

For $n \geq 0$ sufficiently large, some connected subset $\tilde{\sigma}'_n$ of σ_n is at Hausdorff distance at most ϵ from L'_∞ and hence meets any horizontal curve. But this is not possible as, by definition of X , $\tilde{\sigma}_n \subset \tilde{\beta}_n$ is disjoint from the horizontal curve $\tilde{\gamma}$. \square

Proof of Claim 6.10. Fix $n \geq 0$. Let $C_n = [a_n, b_n] = [D^n(a_0), D^n(b_0)]$ be the connected component of the complement in $\mathbb{S}^1 \times \{0\}$ of the minimal Cantor set of D which contains $J \times \{n\}$. As the C_n 's are pairwise disjoint and $\mathbb{S}^1 \times \{0\}$ has finite length, the diameter of C_n tends to 0 as $n \rightarrow +\infty$.

Let $\tilde{D}: \mathbb{R} \rightarrow \mathbb{R}$ be a lift of D to the universal cover of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Recall that the rotation number of D is α . Then it is standard that, for any n ,

$$\tilde{D}^n(\tilde{a}_0) = n\tilde{\alpha} + B_n,$$

where (B_n) is a bounded sequence and \tilde{a}_0 and $\tilde{\alpha}$ are respective lifts of a_0 and α to \mathbb{R} (see [Ghy01, Section 5]). Take a sequence $(k'_n)_n$ of integers such that the sequence $(B_{k'_n})$ converges and $k'_{n+1} - k'_n \rightarrow +\infty$. Observe that the sequence $(D^{k'_{n+1}}(a_0) - D^{k'_n}(a_0) - (k'_{n+1} - k'_n)\alpha)$ converges to 0.

Take k_n sufficiently large so that the set $f^{k_n}(J \times \{0\})$ meets $J \times [k'_{n+1}, +\infty)$. Let $\tilde{\mathcal{L}}_n$ be the line of the universal cover \mathbb{R}^2 of T which joins $(a_{k'_n}, k'_n)$ to $(a_{k'_{n+1}}, k'_{n+1})$ and \tilde{L}_n be either the line which joins $(a_{k'_n}, k'_n)$ to $(a_{k'_n} + (k'_{n+1} - k'_n)\alpha, k'_{n+1})$ or a tiny translate of this line if it meets some lift of the point p . Denote by \mathcal{L}_n and L_n the respective projections of $\tilde{\mathcal{L}}_n$ and \tilde{L}_n to T .

As the sequence $(D^{k'_{n+1}}(a_0) - D^{k'_n}(a_0) - (k'_{n+1} - k'_n)\alpha)$ converges to 0, the Hausdorff distance between \mathcal{L}_n and L_n tends to 0. As the set $f^{k_n}(J \times \{0\})$ meets both $J \times \{0\}$ and $J \times \{k'_{n+1}\}$ and is contained in $J \times \mathbb{R}$, there exists a connected component σ_n of $f^{k_n}(J \times \{0\}) \cap J \times [k'_n, k'_{n+1}]$ which meets both $J \times \{k'_n\}$ and $J \times \{k'_{n+1}\}$. As the diameter of C_n tends to 0 as n tends to $+\infty$, the Hausdorff distance between \mathcal{L}_n and σ_n tends to 0. Finally, the Hausdorff distance between L_n and σ_n tends to 0. \square

One can alternatively show the map f from Construction 6.7 is parabolic with the following argument using bicorns.

Alternative proof of Claim 6.9. Recall we have identified T with $[0, 1] \times [0, 1]$ with the top and bottom edges identified by the Denjoy map D , and the left

and right by rigid translation. Let $\gamma = [0, 1] \times \{0\}$ be the standard horizontal curve on T , let $J \subset \gamma$ be the interval used in the definition of f ; we may parametrize γ so that the left endpoint of J is 0. Choose a horizontal curve $\gamma' = [0, 1] \times \{\epsilon'\}$ slightly above γ so that γ' is transverse to each curve $f^n(\gamma)$ (each iterate will have only countably many points tangent to a horizontal curve, so we may find such a γ' as close as we wish to γ).

Let $J' = [0, \epsilon] \subset J$ be a subinterval on which the bump function in the construction of f is monotone increasing. For each $n > 0$ large enough that $f^n(J')$ intersects γ in at least two points, let $j_n \subset f^n(J')$ be the subarc bounded by the leftmost intersection point on γ of $f^n(J') \cap \gamma'$ (which by construction lies in $[0, \epsilon] \times \{\epsilon'\}$), and the rightmost intersection point of $f^n(J') \cap \gamma'$ on γ' . Let c_n be the union of j_n and the short segment of γ' containing 0 which connects them. This is a closed curve. The isotopy class $[c_n]$ represents a vertex in $\mathcal{C}(T)$; which is simply the Farey graph with vertex set $\mathbb{Q} \cup \{\infty\}$ each point representing the slope of the curve. By construction, as $n \rightarrow \infty$, the average slope of j_n , and hence c_n , approaches α . Thus, necessarily $[c_n]$ eventually leaves each compact set of $\mathcal{C}(T)$.

We now show that this means f cannot act elliptically on $\mathcal{C}^\dagger(T)$. Suppose for contradiction that it did, i.e. $d^\dagger(\gamma, f^n(\gamma)) \leq K$ for some constant $K \geq 0$. Then $d^\dagger(\gamma', f^n(\gamma)) \leq K + 1$. Let P_n be a finite set of points so that each complementary region of $\gamma' \cap f^n(\gamma)$ contains at least one point of P_n . Then γ_n and $f^n(\gamma)$ are in minimal position in $T - P_n$, so by [BHW21, Lemma 3.4] the distance in $\mathcal{C}^s(T - P)$ between $[\gamma_n]_{T - P_n}$ and $[f^n(\gamma)]_{T - P_n}$ is at most $K + 1$ also.

Since c_n is a union of a subarc of γ' and of $f^n(\gamma)$ and these are in minimal position in $T - P_n$, the class $[c_n]_{T - P_n}$ is a bicorn of $[\gamma']_{T - P_n}$ and $[f^n(\gamma)]_{T - P_n}$. In [Ras20] A. Rasmussen proves that the set of bicorns between two curves in the nonseparating curve graph of any finite-type surface satisfies the criterion of Masur–Schleimer [MS13, Theorem 3.5]. In fact any bicorn will be a uniformly bounded distance $L \geq 0$ away from a geodesic [Bow14, Proposition 3.1]. Therefore the distance between $[c_n]_{T - P_n}$ and $[\gamma']_{T - P_n}$ is bounded by at most $K + L + 1$, a bound independent of n . However, there is a 1-coarsely Lipschitz map of the (nonseparating) curve graph of $T - P_n$ to the Farey graph, simply by considering isotopy classes in T . This contradicts our earlier observation that the sequence $[c_n]$ is unbounded in $\mathcal{C}(T)$, a contradiction. We conclude that f acts parabolically on $\mathcal{C}^\dagger(T)$. \square

Remark 6.11. A modification of the construction (by post-composing with the time t_0 -map φ_{t_0} of the flow the homeomorphism f) can be used to show that for any irrational α and any $t_0 \neq 0$ there is a homeomorphism acting parabolically with rotation set $\{(t_0\alpha, t_0)\}$.

Remark 6.12 (Parabolics with irrational slope). A further variation yields homeomorphisms whose rotation sets is a segment of irrational slope through the origin. Simply let $\chi: T \rightarrow [0, 1]$ be a continuous bump function that is *surjective* and vanishes precisely on the suspension of the invariant Cantor

C of the Denjoy map. Then consider the time-1 map of the reparametrized Denjoy flow $\varphi_s(p)$ given by setting $s(t, p) = t(1 - \chi(p))$.

This flow has fixed points and agrees with the Denjoy flow on C . Hence one obtains points $(0, 0)$ and $(1, \alpha)$ in the rotation set. Since f preserves flow lines it is easy to see that all other elements of the rotation set are (positive) multiples of $(1, \alpha)$ and hence $\rho(f)$ is a segment of irrational slope.

6.3. Proof of Theorem 1.1. A similar construction can also be used to build homeomorphisms of higher genus surfaces S which act parabolically on $\mathcal{C}^\dagger(S)$, proving Theorem 1.1. We give the details now, and discuss alternative constructions below.

Proof of Theorem 1.1, first construction. Let S be a surface of genus $g \geq 2$, fix a hyperbolic structure on S , and let Λ be a minimal filling geodesic lamination on S . Let X be a vector field supported on the complement of Λ , so that the flow of X pushes points into the cusps of the lamination. This may be defined explicitly on each complementary region of Λ , modeled on a vertical flow supported on a standard ideal hyperbolic triangle with one vertex at infinity in the upper half plane. Let f be the time-one map of this flow. Note that by cutting off the support of the flow, we may view f as a C^0 -limit of homeomorphisms supported on disks.

Let $\hat{S} \rightarrow S$ be a finite cover of S . Via pullback, \hat{S} inherits a hyperbolic structure from S , and the preimage $\hat{\Lambda}$ is a minimal filling geodesic lamination of \hat{S} . We may lift our flow described above to \hat{S} and write \hat{f} for the time-one map. The minimality of $\hat{\Lambda}$ ensures the following: for any simple closed curves γ and γ' of \hat{S} , there exists N such that for any $n > N$ we have that $\hat{f}^n \gamma$ intersects γ' . This is because γ intersects the support of \hat{f} (every half-leaf is dense in $\hat{\Lambda}$), moreover there is some subarc c of γ inside the support that connects different half-leaves of $\hat{\Lambda}$. Deeper into the cusp, there is a subarc c' of γ' likewise connecting different half-leaves. By definition of \hat{f} , eventually $\hat{f}^n c$ intersects c' for sufficiently large n .

It thus follows from Lemma 4.3 that any f -orbit in $\mathcal{C}^\dagger(S)$ has infinite diameter and so f is not elliptic. On the other hand, f is the C^0 -limit of disk-supported maps. Since such maps act elliptically on $\mathcal{C}^\dagger(S)$, C^0 -continuity of the asymptotic translation length (Theorem 1.2) shows that f also has translation length 0. This shows that it is parabolic. \square

Proof of Theorem 1.1, second construction. Start with an abelian differential on S and pick a slope λ that defines a minimal (oriented) foliation \mathcal{F}_λ . The abelian differential endows S with a translation surface structure, in particular a metric with zero curvature off the singularities, which we normalise with unit area. For simplicity we will assume that there are no saddle connections of slope λ .

Consider a smooth vector field on S with direction λ with zeroes occurring only at the singularities of the abelian differential. Let f be the time-1 map of the flow along the vector field.

We now prove that f does not act hyperbolically on $\mathcal{C}^\dagger(S)$. To do this, we will find for any $n \in \mathbb{N}$ a curve α such that $d^\dagger(\alpha, f^n\alpha) \leq 2$. Given $n \in \mathbb{N}$, pick a (small) closed interval J perpendicular to the slope λ , such that $\cup_{t \in [0, n]} \phi_t J$ is disjoint from the singularities, and such that the $\phi_t J$ are pairwise disjoint. The existence of such an interval follows from the fact that \mathcal{F}_λ contains an infinite leaf. Choose α to be a simple closed curve which is a union of an interval of slope λ , and a subinterval of J (which is possible because there is a half-leaf of slope λ through J which is dense). Observe that the complement of $\alpha \cup f^n\alpha$ consists of two regions, one of which is a rectangle, foliated by the $\phi_t J$. It must be the case that the other region carries an essential simple closed curve disjoint from $\alpha \cup f^n\alpha$, and so we conclude that $d^\dagger(\alpha, f^n\alpha) \leq 2$ and thus f is not hyperbolic.

To finish the proof we now prove that f is not elliptic, by considering a curve containing a horizontal segment adjacent to a singularity. For concreteness, define a curve α on S by starting at a singularity x , following a line of slope λ until one intersects a horizontal through the same singularity, then following that horizontal to x in order to close the curve. Thus, α is the union of a horizontal segment τ , and a segment of slope λ each of which have one endpoint at x . We claim that $d^\dagger(\alpha, f^n\alpha)$ is unbounded. To show this, suppose for contradiction that $d^\dagger(\alpha, f^n\alpha) \leq K$, for all $n \in \mathbb{Z}$. By Lemma 4.3 there is a finite-sheeted cover \hat{S} of S such that α and $f^n\alpha$ admit disjoint elevations, for any $n \in \mathbb{Z}$.

Let μ be the half-leaf of \mathcal{F}_λ emanating from x . Note that every lift of μ to S is dense in S . As n increases, $f^n(\tau)$ converges on compact sets to μ . For n sufficiently large, any elevation of $f^n(\tau)$ to S will intersect every horizontal segment of fixed length, and thus intersects any elevation of τ . This contradicts that α and $f^n\alpha$ should have disjoint elevations, concluding the proof. \square

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