Laplacians on the Sierpinski Gasket Julia set

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Abstract

The study of Julia sets gives a new and natural way to look at fractals. When mathematicians investigated the special class of Misiurewicz's rational maps, they found out that there is a Julia set which is homeomorphic to a well known fractal, the Sierpinksi gasket. Here we apply the methods of defining an Laplacian developed by Bob Strichartz et al. to give rise to another construction on SG with an inherited dynamical behavior.

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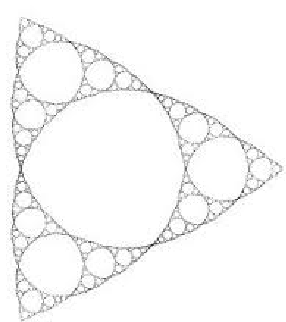

Figure 1: The Julia set of $z^2 - 0.59267/z$

1 Julia sets of rational maps

Complex dynamics is not restricted to polynomial maps. One can also investigate the dynamics of rational functions $R(z) = \frac{P(z)}{Q(z)}$ where P and Q are complex valued polynomials. One might react critical to the case when z is a root of Q, since R is mapped to ∞, but for the iteration of functions, ∞ is not a special point. Hence, one deals with maps $R : \mathbb{P}^1 \mapsto \mathbb{P}^1$, where $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. Indeed $\mathbb{P}^1$ can be identified as the usual 2-sphere which also provides a metric.

Now one wants to define a Julia set for the rational map. The definition for polynomials by bounded orbits does not work here anymore. Instead, one defines Julia sets by normal families. The definition and further mentioned properties are from [Bla84], who gives a rigorous introduction into the dynamics of rational maps.

Definition 1. [1] *Let U be an open subset of $\mathbb{P}^1$ and $\mathsf{F} = \{f_i | i \in I\}$ a family of meromorphic functions on $\mathbb{P}^1$ defined on U (I is any index set). The family F is a normal family if every sequence f_n contains a subsequence f_{n_j} which converges uniformly on compact subsets of U.*

Definition 2. *The Fatou set $\mathcal{F}$ of a rational map $R : \mathbb{P}^1 \mapsto \mathbb{P}^1$ is the set of points that have a neighborhood on which the sequence of iterates R^n forms a normal family. The Julia set $\mathcal{J}$ is the set of points that have no such neighbourhood.*

The definition might look a bit awkward, but it coincides with the definition for polynomials and similar properties of Julia sets still hold: The Julia set is compact and completely invariant, meaning that

$$R(J) = J = R^{-1}(J). \tag{1}$$

The formal definition of a Julia set is not intrinsically useful to decide, whether a point belongs to the Julia set or not. For periodic points this can be decided rather easily, and together with the invariance property one can conclude for more points to be in the Julia set.

Definition 3 ([Bla84] p.93)**.** *The periodic orbit $O+(z_0)$ of a periodic point consists of all points $R^k(z_0)$ for $1 \leq k < n$ and $R^n(z_0) = z_0$. Let $\mu = (R^n)'(z_0)$. A periodic orbit is:*

- *attracting if* $0 < |\mu| < 1$
- *superattracting if* $\mu = 0$
- *repelling if* $|\mu| > 1$

[1] [Bla84] p.89

- *indifferent if* $|\mu| = 1$

Proposition 1. *If $O^+(z_o)$ is a (super)attracting periodic orbit, then it is contained in $\mathcal{F}$. If it is a repelling orbit, then it is contained in $\mathcal{J}$.*

One should note that μ is a constant for the orbit independent of the choice of z_0, this can be seen by repeatedly applying the chain rule:

$$(R^n)'(z_0) = R'(R(z_0)) \cdot R'(R^2(z_0)) \cdots R'(R^n(z_0)) \tag{2}$$

A class of rational maps we will focus on are so called *Misiurewicz* rational maps. They are defined by the special properties of its critical values, i.e. all points $z \in \mathbb{C}$ satisfying $R'(z) = 0$, which always play an essential role to understand the dynamics. Denote the set of critical values by $CV(R)$. And call $\Omega(R)$ the ω-limit set of $CV(R)$, that means $z \in \Omega(R)$ iff there exists a $c \in CV(R)$ and an unbounded sequence n_k of positive integers such that $z = \lim_{k\to\infty} R^{n_k}(c)$. Finally, let $\omega(R) := \Omega(R) \cap \mathcal{J}$.

Definition 4 ([DU91] p.200)**.** *A rational map R is Misiurewicz or subexpanding if $R|_{\omega(R)}$ is expanding:*

$$\exists s \geq 1\, \exists \mu' > 1\, \forall z \in \omega(R): \quad |(R^s)'(z)| \geq \mu'$$

.

2 Dynamics on SG

In [DRS06] one investigates rational maps of the form $z^n + \frac{\lambda}{z^m}$ with gasket like Julia sets. A generalized Sierpinksi gasket is described as having a N-fold symmetry and from the second stage and onward of the construction, m corners of a removed region lie in the boundary of one of the removed regions in the previous stage, with $1 \leq m < N$. It is proven that the structure of a generalized SG for those described maps occurs, when they are so called MS-maps:

Definition 5 ([DRS06] Def. 2.2)**.** *A map is called Misiurewicz-Sierpinski map or just MS-map if*

- *each critical point lies in the boundary of the immediate basin of infinity*
- *each of the critical points is preperiodic*

One should note that a MS-map is always Misiurewicz:

Proposition 2. *If all critical points of a rational map R are preperiodic (without indifferent periodic points), then it is Misiurewicz.*

Proof. Let $c_1, ..., c_n$ be the critical points of R, who reach a cycle of periods $p_1, ..p_n$ and elements of the orbits $z_1, .., z_n$, respectively. The union of the orbits will be the ω-limit set. All points contained in $\omega(R)$ will satisfy $|(R_i^p)'(z_i)| > 1$. Let $s = \prod_{i=1}^n c_i$. Similar to (2) one has:

$$(R^s)'(z_i) = \Big(R'(R(z_i)) \cdot R'(R^2(z_i)) \cdots R'(R^n(z_i))\Big)^{s/p_i}$$

All of these derivatives will have absolute value greater than one, take μ' to be the minimum of them. $\square$

Now to describe the symbolic dynamics, let β_λ be the boundary of the basin of infinity and τ_λ the boundary of the neighbourhood of 0 that is mapped to the basin of infinity, also called trap door. The critical points are now exactly the intersection points of β_λ and τ_λ. Moreover, define $\tau_\lambda^k = R^{-k}(\tau_\lambda)$ which consists of several connected components and are the removed regions from the second step and onward of the construction of the Julia set. The Julia set is now the closure of $\beta_\lambda \cup \bigcup_{k\geq 0} \tau_\lambda^k$.

From now on we will focus on the case $R(z) = z^2 + \frac{\lambda}{z}$ with $\lambda = -\frac{16}{27}$ where the resulting Julia set is homeomorphic to the standard Sierpinski Gasket. The map has three critical points $c_0 = -2/3$, $c_1 = 1/3 + 0.577i$ and $c_{\bar{1}} = 1/3 - 0.577i$. The point $z_0 = R(c_0) = 4/3$ is a fixed point with $R'(z_0) = 3$, hence it lies on the Julia set. Moreover, $z_1 = R(c_1)$ and $z_{\bar{1}} = R(c_{\bar{1}})$ form a 2-periodic cycle and $(R^2)'(z_1) = (R^2)'(z_{\bar{1}}) = (R^2)'(z_0) = 9$. Hence, by Proposition 2 the map is Misiurewicz. The outer triangle of SG with vertices $z_0, z_1, z_{\bar{1}}$ corresponds to β_λ and the first removed triangle in the center of SG has vertices $c_0, c_1, c_{\bar{1}}$. At the next step τ_λ^1 consists of the three smaller triangles removed in the second step of the construction of SG. And exactly $m = 1$ corners of these removed regions lie in the boundary of the removed region in the previous stage.

This gives rise to a completely new construction method of SG with a dynamical background. One takes the same graph approximation defined by $V_0 = \{z_0, z_1, z_{\bar{1}}\}$ and $V_{m+1} = R^{-1} V_m$, but with a completely different mapping. The mapping for V_2 as an example is shown in Figure 2.

3 Standard Energy

As developed in [Str06] since one has now a graph approximation one wants to construct a Laplacian on it, the first step is to define a graph energy:

$$E_m(u,v) = \sum_{x \sim_m y} c_m(x,y)(u(x)-u(y))(v(x)-v(y)) \quad x,y \in V_m$$

In order to respect the dynamics, we want to choose suitable conductances $c_m(x,y)$ such that the Energy is invariant:

$$E_m(u \circ R, v \circ R) = c \cdot E_{m-1}(u,v) \tag{3}$$

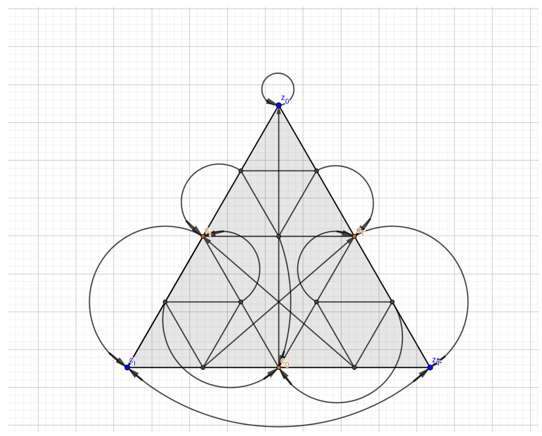

Figure 2: Dynamics for V_2

with some constant c independent of u, v or m. For $m = 1$ one has for $u = v$ [2]:

$$E_0(u) = c(z_0, z_1)(u(z_0) - u(z_1))^2 + c(z_1, z_{\bar{1}})(u(z_1) - u(z_{\bar{1}}))^2 + c(z_{\bar{1}}, z_0)(u(z_{\bar{1}}) - u(z_0))^2$$

and

$$\begin{aligned} E_1(u \circ P) = &(c(z_1, c_1) + c(c_1, c_{\bar{1}}) + c(c_{\bar{1}}, z_{\bar{1}}))(u(z_1) - u(z_{\bar{1}}))^2 \\ &+ (c(c_1, z_0) + c(c_0, z_{\bar{1}}) + c(c_0, c_1))(u(z_0) - u(z_1))^2 \\ &+ (c(z_0, c_{\bar{1}}) + c(c_0, z_1) + c(c_0, c_{\bar{1}}))(u(z_{\bar{1}}) - u(z_0))^2. \end{aligned} \tag{4}$$

There are multiple solutions for the conductances such that the invariance property is fulfilled, the easiest solution would be to set all conductances to 1. For higher levels, this will still satisfy (3) since the degree of the map is 3, thus each point has 3 preimages and always three of the 3^{m+1} edges in V_m are identified. Hence, one obtains the identity:

$$E_m(u \circ R^k, v \circ R^k) = 3^k E_{m-k}(u, v) \tag{5}$$

The graph energy is not different from the standard energy for SG. So the energy renormalization is the same:

$$\varepsilon_m = \left(\frac{3}{5}\right)^{-m} E_m(u) \tag{6}$$

Combining this equation with (5) and passing m to infinity one obtains an invariant energy:

$$\varepsilon(u \circ R^k, v \circ R^k) = 5^k \varepsilon(u, v) \tag{7}$$

[2] the invariance property still holds for the bilinear form by the polarization identity, see [Str06] Eq. (1.3.3)

4 Measure and Laplacian

The next step is to define a suitable measure on SG.Again to respect the dynamics the measure should be invariant:

$$\mu(A) = \mu(R^{-1}A)$$

for every Borel set A and therefore

$$\int_A f \circ R d\mu = \int_A f d\mu \tag{8}$$

.

A theorem by Denker and Urbański states that there is not much choice:

Theorem 1 ([DU91]Theorem 4.6)**.** *For a Misiurewicz rational map R there exists a unique, ergodic, R-invariant probability measure μ.*

The following Lemmata show now that the standard measure on SG is the only suitable measure.

Lemma 1. *The standard measure μ is R-invariant.*

Proof. Every Borel set A can be approximated arbitrarily well by a finite union of cells in SG, say $A \approx \bigcup F_i$ where the F_i are level m cells. By the continuity of R it suffices to show the invariance property for this set. The preimage of every m-cell consists of three $m+1$ cells. Since a $m+1$ cell has one third of the measure of the next higher level cell, one may conclude

$$\mu(\bigcup F_i) = \mu(\bigcup R^{-1}F_i)$$

□

Lemma 2. *The standard measure μ is ergodic.*

Proof. Two measures are said to be equivalent if they share the same sets of measure zero. The invariant measure μ_i from [DU91] Thm. 4.6 is equivalent to the h-conformal measure μ_ν which is said to be equivalent to the Hausdorff measure [3]. Finally, the Hausdorff measure on SG is equivalent to the standard measure μ_s [4]. Recall that ergodicity means that all sets satisfying $A = R^{-1}A$ have full measure or are zero sets. Now μ_i is defined to be ergodic and equivalent to μ_s. Hence, for every invariant set A if $\mu_i(A) = 0$ then $\mu_s(A) = 0$ and if $\mu_i(A) = 1$ then $\mu_i(A^c) = \mu_s(A^c) = 0$ and hence $\mu_s(A) = 1$. □

Thus one obtains the standard Laplacian, defined by the weak formulation:

$$-\varepsilon(u,v) = \int (\Delta_\mu u) v \, d\mu \quad \forall\, v \in dom\, \varepsilon. \tag{9}$$

[3] [DU91] p.195

[4] [Str06] p.6

Combining (3),(8) and (9) one obtains:

$$\Delta_\mu(u \circ R^k) = 5^k \Delta_\mu(u). \tag{10}$$

It immediately follows a property about the eigenfunctions of the operator:

Theorem 2. *If u is an eigenfunction of Δ_μ with eigenvalue λ, then $u \circ R^n$ is also an eigenfunction of Δ_μ with eigenvalue $5^k\lambda$. In particular, $5\Sigma_\mu \subset \Sigma$, where Σ_μ is the spectrum of Δ_μ.*

5 Iterated Function System

It can be observed that the map R has a linear behaviour in the region formed by the triangle $c_1, z_0, c_{\bar{1}}$, that we will name F_0K. Treating with the linear maps on SG F_0, F_1, F_2 as discussed in [Str06], for points $z \in F_0K$ one has in a combinatoric sense

$$R(z) = F_0^{-1}(z) \tag{11}$$

One should mention that the Julia set in figure 1 is not exactly formed by triangles, because some edges are concave and one does not have a contraction by the factor $\frac{1}{2}$ everywhere as in SG. But for the combinatoric purposes of defining graph energies, this does not matter.

Moreover, all maps of the form $z^2 + \frac{\lambda}{z}$ have the symmetry properties:

$$R(\omega z) = \omega^2 R(z) \tag{12}$$

$$R(\omega^2 z) = \omega R(z) \tag{13}$$

where ω is the rotation of a third circle, $\omega = e^{\frac{2}{3}\pi i}$. Note that one can express the maps F_1 and F_2 in terms of F_0 and rotations:

$$F_1^{-1} = \omega \circ F_0^{-1} \circ \omega^2 \tag{14}$$

$$F_2^{-1} = \omega^2 \circ F_0^{-1} \circ \omega \tag{15}$$

Iff $z \in F_0K$ then $\tilde{z} := \omega z$ lies in F_1K. Now (12) becomes

$$R(\tilde{z}) = \omega^2 R(\omega^2 \tilde{z})$$

Since $\omega^2 \tilde{z} = z \in F_0K$, one may apply (11) to get:

$$R(\tilde{z}) = \omega^2 \circ F_0^{-1} \circ \omega^2(\tilde{z})$$

And (14) yields:

$$R(\tilde{z}) = \omega \circ F_1^{-1}(\tilde{z}) \tag{16}$$

Similar, for $\tilde{\tilde{z}} := \omega^2 z \in F_2K$ equation (13) becomes:

$$R(\tilde{\tilde{z}}) = \omega F_0^{-1}(\omega \tilde{\tilde{z}}) = \omega^2 \circ \omega^2 \circ F_0^{-1} \circ \omega(\tilde{\tilde{z}})$$

And (15) yields:

$$R(\tilde{\tilde{z}}) = \omega^2 \circ F_2^{-1}(\tilde{\tilde{z}}) \tag{17}$$

And for the preimage R^{-1} equations (11), (16), (17) become:

$$R^{-1}|_{F_0K} = F_0 \tag{18}$$

$$R^{-1}|_{F_1K} = F_1 \circ \omega^2 \tag{19}$$

$$R^{-1}|_{F_2K} = F_2 \circ \omega \tag{20}$$

Hence, one obtains the iterated function system (IFS):

$$K = R^{-1}(K) = \bigcup_{i=0}^{2} \tilde{F}_i K \tag{21}$$

where $\tilde{F}_0 = F_0, \tilde{F}_1 = F_1 \circ \omega^2$ and $\tilde{F}_2 = F_2 \circ \omega$.

Furthermore, one can replace K by the graph approximation V_{m+1} in the inverse formulas (18) – (20):

$$V_{m+1} = R^{-1}(V_m) = \bigcup_{i=0}^{2} \tilde{F}_i V_m \tag{22}$$

6 Self-Similar Energy Forms

We seek a regular Dirichlet form ε that satisfies the self similar identity:

$$\varepsilon(u) = \sum_i r_i^{-1} \varepsilon(u \circ \tilde{F}_i) \tag{23}$$

for a set of weights $\{r_i\}$ satisfying $0 < r_i < 1$. Equivalently, one seeks a solution to the renormalization problem. Given initial positive conductances on v_0 one defines the energy on V_1 by

$$\varepsilon_1(u) = \sum_{x \sim_1 y} c_1(x,y)(u(x) - u(y))^2 \tag{24}$$

for

$$c_1(\tilde{F}_i x, \tilde{F}_i y) = r_i^{-1} c(x,y) \quad \text{if } x, y \in V_0.$$

Note that this implies the energy invariance (3) for $c = r_0^{-1} + r_1^{-1} + r_2^{-1}$. One says that ε_0 solves the renormalization problem with given weights if

$$\varepsilon_1(\tilde{u}) = \lambda \varepsilon_0(u) \tag{25}$$

for the harmonic extension $\tilde{u}$ on V_1. After finding out the constant λ, one renormalizes the weights by setting $\tilde{r}_i = \lambda^{-1} r_i$. Then the graph energy can be defined in a similar sense for higher levels which converge to an energy on $K = SG$. This problem has been studied for the standard IFS and for SG with

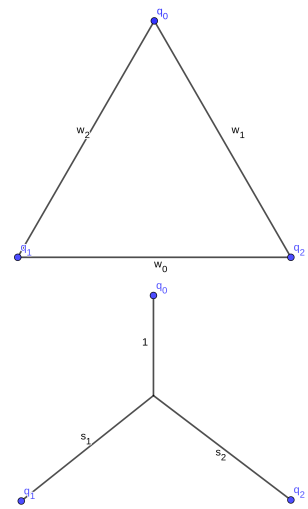

Figure 3: Network transformation on V_0

twists [CS07]. Here we will investigate if one can find an equally simple answer for our case involving rotations. In order to determine the existence of λ in (25) one uses the electric network interpretation and $\Delta - Y$ transformations [5]. Let c_0, c_1, c_2 be the initial conductances on V_0. Denote $w_i = c_i^{-1}$ the initial resistances on V_0. Since the weights are renormalized afterwards anyway, one may set $r_0 = 1$.

In Figure 3 the $\Delta - Y$ transformation is shown for V_0. Without loss of generality, one may set $\frac{w_1 w_2}{D} = 1$ and denote $s_1 = \frac{w_0 w_2}{D}$ and $s_2 = \frac{w_0 w_1}{D}$ where $D = w_0 + w_1 + w_2$.

[5][Str06] pp. 23-27

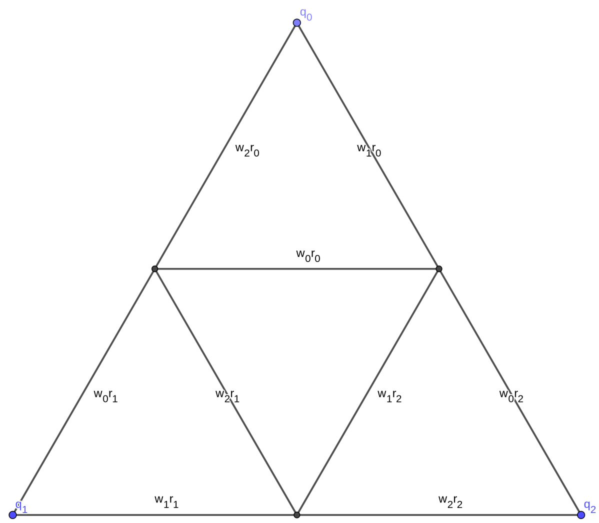

Figure 4: Resistances on V_1

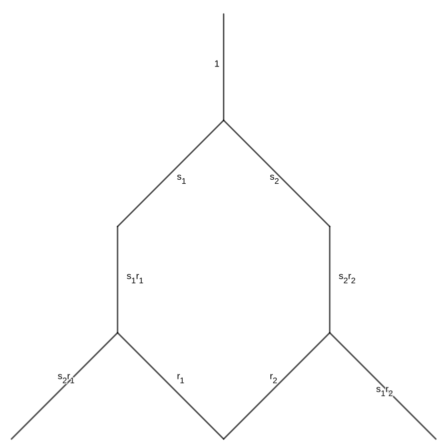

Figure 5: First step of network transformation

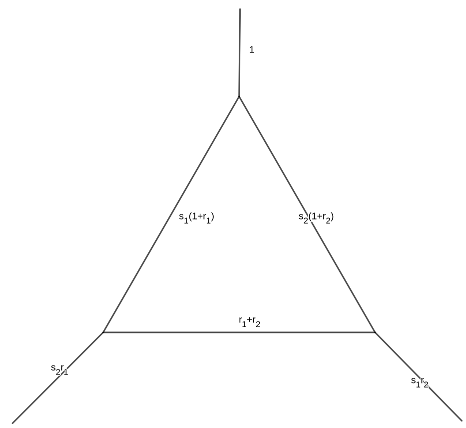

Figure 6: Second step of network transformation

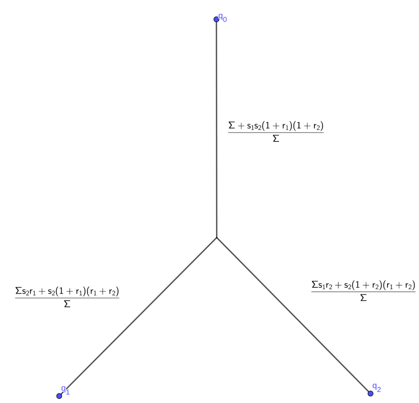

Figure 7: Final step of network transformation

Corresponding to (24), the resistances on V_1 are shown in Figure 4. Now one applies $\Delta - Y$ transformations to each of the three triangles and expresses the result in terms of s_1 and s_2, see Figure 5. After summing up adjacent resistances one obtains Figure 6. A last transformation and summing up the resistances at the outer edges finally yields to Figure 7. For simplicity, abbreviate $\Sigma = r_1 + r_2 + s_1 + s_2 + s_1 r_1 + s_2 r_2$. The resulting network should be now a multiple of the Y-network in Figure 3 in accordance to (25). Hence, one obtains the system of equations:

$$\Sigma + s_1 s_2 (1+r_1)(1+r_2) = \lambda \Sigma \tag{26}$$

$$\Sigma s_2 r_1 + s_1 (1+r_1)(r_1+r_2) = \lambda s_1 \Sigma \tag{27}$$

$$\Sigma s_1 r_2 + s_2 (1+r_2)(r_1+r_2) = \lambda s_2 \Sigma \tag{28}$$

One can use (26) to determine λ. Since the weights and conductances are positive, $\lambda > 1$ and so $\tilde{r}_0 = \lambda^{-1} < 1$. The remaining equations are just:

$$\Sigma s_2 r_1 + s_1 (1+r_1)(r_1+r_2) = \Sigma s_1 + (1+r_1)(1+r_2) s_1^2 s_2 \tag{29}$$

$$\Sigma s_1 r_2 + s_2 (1+r_2)(r_1+r_2) = \Sigma s_2 + (1+r_1)(1+r_2) s_1 s_2^2 \tag{30}$$

The essential observation is that (29) is linear in r_2 and (30) is linear in r_1.This is not a trivial fact and only occurs on lucky cases of an IFS like SG with twists. This shows once more, that complex dynamics are the right way to look at it.

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