THE RELATIONSHIP BETWEEN THE CRITICAL SETS AND MORSE CODES OF ELECTRICALLY CHARGED KNOTS

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ABSTRACT. Consider a knot $K$ in $S^3$ with uniformly distributed electric charge. From the standpoint of both physics and knot theory, it is natural to try to understand the critical points of the potential and their behavior.

By taking successive preimages of regular potential values, we get an $N$–tuple of compact orientable surfaces, whose genera we define as the Morse code. We relate the topological data of the critical set to the Morse code. We show that critical points of index 1 correspond to increases in successive terms in the Morse code, whilst critical points of index 2 correspond to decreases. We conclude with a description of the bifurcation of critical points under generic knot isotopies. Our theorems are proven with Morse theory and techniques from geometric topology.

keywords: knot theory, electrostatics, Morse theory, Morse code, dynamical systems, geometric topology, Cerf theory

1. Introduction

Imagine a closed knotted wire in space fixed in place, with uniform electric charge distributed on it. Our novel problem of interest is to understand the dynamics of the knot’s electric field, which was originally motivated by applications to the study of knotted DNA molecules [8] in viruses, the design of electrical circuits [2], and broader questions in physical knot theory [4]. However, the pure mathematical phenomena found in electromagnetic knot theory is not well understood.

The electric field is the negative gradient flow of a potential function, so the zeros of the electric field correspond to critical points of the potential. Thus some basic questions arise: How many critical points are there? What are their local behaviors? And how does the critical set relate to topological properties of the knot?

Let $K \subseteq \mathbb{R}^3 \subseteq S^3$ be a smooth knot parametrized by the curve $r(t)$, $t \in [0, 2\pi]$ with $r(0) = r(2\pi)$. We will take the convention that $S^3$ is the union of $\mathbb{R}^3$ and a single compactifying point at infinity. Suppose $K$ is endowed with a uniform charge distribution. With a choice of units, the electric potential between a point $k \in K$ and a point charge at $x$ at a distance $R$ from $k$ is proportional to $R^{-1}$ by Coulomb’s Law. It therefore makes sense to define the electric potential $\Phi : S^3 - K \to \mathbb{R}$, on the complement of $K$ by the line integral

\[ \Phi(x) = \int_{k \in K} \frac{dk}{|x - k|} = \int_0^{2\pi} \frac{|r'(t)|}{|x - r(t)|} dt, \quad x \in \mathbb{R}^3 - K. \]
We set $\Phi(\infty) = 0$ to ensure smoothness. By differentiating under the integral sign with respect to $x$, we can see the electric potential is smooth and harmonic. The electric field is defined by $E = -\nabla \Phi$. The critical points of $\Phi$ represent equilibrium points where a charged particle at rest will continue to experience no electric force from the charge distribution. Some conventions define the potential to be negative, in order for the electric field to point towards the knot, but it is more convenient for our purposes to work with a nonnegative potential function.

The electric potential is difficult to analyze in general, as integral formulas can be highly nonlinear and nonsymmetric, but the potential and electric field integrals can be numerically approximated via Gaussian quadrature, and the solutions to $E(x) = 0$ can be determined by the multivariable Newton method. Initial numerical approximations of the critical sets for various knots were coded by Townsend and Lipton in 2018, showing that there are isotopic parametrizations of the unknots which have critical sets of differing size. In particular, the trivial unknot parametrization $r(t) = (\cos(t), \sin(t), 0)$ has a single critical point in the origin, whilst the $(5, 1)$ torus knot with parametrization $r(t) = (\cos(t + 2) \cos(5t), \cos(t + 2) \sin(5t), -\sin(t))$ has three critical points, as depicted in Figure 1.

![Figure 1. A top-down projection of two knots isotopic to the unknot with differing critical sets.](image)

It is unsurprising that the critical set is not invariant under isotopy, given the dependence of $\Phi$ on $r$, but a 2019 result of Lipton [6] proved there is indeed a restriction on the critical set based on a knot invariant called the tunnel number. Let $cp(K)$ denote the smallest size of the critical set of $K$ over all parametrizations, which includes the critical point at infinity in the count. Let $t(K)$ denote the tunnel number, which is the smallest number of arcs one must add to $K$ such that the complement is a handlebody. The aforementioned result is stated as follows.

**Theorem 1.** For all knots $K$, $cp(K) \geq 2t(K) + 2$.

The result was proven using techniques in geometric topology, Morse theory, and stable manifold theory. The proof does not apply Morse reconstruction to $\Phi$ itself, but rather, it applies the Morse Rearrangement Lemma (see Lemma 2.4.12 of [7]) to $\Phi$, which allows us to alter the global, long-term dynamical behavior of the gradient to conform to certain convenient properties, whilst leaving the local behavior around the critical points unchanged.

The Morse-theoretic behavior of the true electric potential had yet to be examined. The critical values of $\Phi$ partition the positive real numbers into finitely many intervals, and a key idea from Morse theory states that the level sets of two values from the same interval are homeomorphic. This leads us to the central object of this article: the Morse code of a knot, which is a tuple of numbers listing the genera of the equipotential surfaces corresponding to each of the intervals. Another key idea from Morse theory states the passage from one equipotential surface to another is determined, up to homotopy equivalence, by the attachment of a disc of appropriate dimension. However, these attachment maps can be quite pathological in general, so we needed to compute some examples in order to anticipate the properties of Morse codes before we could proceed with a mathematically rigorous proof.
The result of this article describes the general relationship between Morse codes and critical sets. We saw through several computational approximations, described in Section 3, that critical points of index 1 correspond to an increase in genus, whilst critical points of index 2 correspond to a decrease in genus. We were able to prove this conjecture using methods from geometric and algebraic topology. Finally, we applied Cerf theory to describe the bifurcation of critical points under knot isotopy, and we proved almost all knot isotopies (under the $C^\infty$ topology) induce saddle-node bifurcations, creating or destroying critical points in pairs.

2. Preliminary definitions

In this section, we formally define the notion of a “Morse code” of an embedded knot. The Morse code, whilst difficult to compute with mathematical certainty, is readily computable in practice with visualization software.

Recall a Morse function $f$ is a smooth, real-valued function whose critical points are nondegenerate (the Hessians at the critical points are nonsingular). Let $\text{Crit}(f) = \{p_0, \ldots, p_N\}$ denote the critical set of a Morse function. The index of a critical point is the number of negative eigenvalues of its corresponding Hessian. These properties are invariant under change of coordinates.

Definition 1. A critical set is distinct (or alternatively, nonresonant) when each of the critical values $f(p_0), \ldots, f(p_N)$ are distinct.

Let $\Phi$ be the electric potential of the knot $K$, with a given parametrization. Consider $\text{Crit}(\Phi)$ and label the points $p_0, p_1, \ldots, p_N$ such that $p_0 = \infty$, and $\Phi(p_i) \leq \Phi(p_j)$ whenever $i \leq j$, and let $V_0 = 0 < V_1 < \cdots < V_N'$ be the distinct critical values of $\Phi$. Note that unless the critical set is distinct, we do not necessarily have that $\Phi(p_i) = V_i$ for all $i$.

Let $G_i$ be the genus of the surface $S_i = \Phi^{-1}(V_i + \varepsilon)$ where $\varepsilon > 0$ is chosen such that $V_i + \varepsilon_0$ is a regular value for each $0 < \varepsilon_0 \leq \varepsilon$. Because there are only finitely many $i$, a single $\varepsilon$ can be chosen to work for all $i$.

By the implicit function theorem, each $S_i$ is a smooth, compact, orientable surface without boundary embedded in $\mathbb{R}^3$. By the Morse Reconstruction Lemma [7] the topology of each $S_i$ is fixed for properly chosen $\varepsilon$, and thus each $G_i$ is well-defined.

Definition 2. The Morse code of $K$ is the $N'$-tuple $(G_0, G_1, \ldots, G_{N'})$. A Morse code is distinct when the critical set is distinct.

Remark: Clearly, for a distinct critical set, $N = N'$.

Remark: Not all critical sets and Morse codes are distinct, as will be shown in the example of the trefoil with order 3 rotational symmetry. Distinctness is a generic property, however. By introducing a perturbation in the knot, we can change a nondistinct Morse code into a distinct Morse code.

There are some lemmas that follow immediately from the definitions and from well-known theorems.

Lemma 1. All Morse codes begin with 0 and end with 1.

Proof. The first statement follows from a multipole expansion sufficiently far away from the origin. The second statement follows from the theory of conductive matter. See Chapters 2.5 and 3.4 of Griffiths [3].
Intuitively, this means the equipotential surface for a very small voltage level will be a large topological spheroid surrounding the knot, and the equipotential surface for a very high voltage level will be a thin tube surrounding the knot.

**Example:** The most basic example of a Morse code comes from the trivial embedding of the unknot, defined by \( r(t) = (\cos t, \sin t, 0) \). We can immediately see that any finite critical point must lie on the \( z = 0 \) plane, as otherwise, the \( z \)-coordinate of the electric field will be the integral of a continuous strictly nonzero function divided by a strictly positive function, which is nonzero. We can then see that the origin is a critical point by symmetry, as the force exerted on the origin by any point on the knot is cancelled by an opposing force on the plane reflected through the origin. It is a routine calculation to see that there are no other critical points on the plane, as seen in Zypman \([9]\).

This critical point has index 1, as a perturbation in the \( z = 0 \) plane will send a test charge towards the knot, whilst a perturbation along the \( z \)-axis will send a test charge back towards the origin, as all attracting charge is concentrated in the \( z = 0 \) plane. As for the Morse code, there are only two topologically equipotential surfaces, so by Lemma 1, the Morse code is \((0, 1)\).

**Lemma 2.** Let \( m_i \) denote the number of critical points which have index \( i \). Then \( m_1 - m_2 = 1 \) and \( m_1 + m_2 + 1 = N \).

**Proof.** This is a consequence of the Morse inequalities. See Lipton \([6]\). \(\square\)

**Example:** Returning to the trefoil example, the order 3 symmetry of the knot implies the origin is a critical point. Suppose we take for granted that there is one other finite critical point that is not the origin. By symmetry, extra critical points come in triplets, each having the same critical value. This gives us a count so far of \( N = 5 \) critical points (including \( \infty \)). However, there are no integer solutions \((m_1, m_2)\) for the system of equations given in Lemma 2, which means \( N \) is larger. The next candidate for \( N \) is \( N = 8 \), which has the solution \( m_1 = 4, m_2 = 3 \). The numerical computations in the next section suggests this is indeed the size of the critical set for a given parametrization of the trefoil.

### 3. Computational Approximations

The code used in this section was written in Python and is publicly available on a GitHub repository \([5]\). This code includes parametrizations for several types of knots. In this section, we will describe our full methodology for computing the critical set and Morse code for a trefoil, followed by some additional imagery which indicates the Morse code for two other, more complex knot types.

![Figure 2. Two views of the trefoil knot, visualized in Matplotlib.](image-url)
Consider the following parametrization of a trefoil, depicted in Figure 2:

\[
\mathbf{r}(t) = (\sin t + 2 \sin 2t, \cos t - 2 \cos 2t, -\sin 3t).
\]

We can evaluate the potential and the appropriate derivatives by the Gaussian quadrature approximation of an integral. The implementation of the multivariable Newton method requires us to compute the Jacobian of \(E\) for each iteration. In general, this would mean computing nine partial derivatives, but since the Jacobian is a Hessian of a harmonic function, we can reduce computation time because we need only compute five distinct numerical integrals. By setting initial conditions on a mesh grid surrounding the knot, we get the output depicted in Figures 3 and 4.

We can then compute visualizations of the equipotential surfaces for four different regular values spaced between the three critical values. We visualize the surfaces \(\Phi(x) = 12, 13, 15.6, 16\) by the Marching Cubes algorithm. By rotating and zooming in on a surface visualization, we can compute the genus, which is often not immediately obvious for equipotential surfaces for more complex knots.

As seen in Figure 5, the Morse code for this trefoil is \((0, 3, 4, 1)\). It would seem that crossing a critical value corresponding to a critical point of index 1 corresponds to an increase in genus by 1 for each critical point, whilst crossing a critical point of index 2 corresponds to a decrease in genus by 1 for each critical point.

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**Figure 5.** The Morse code for this trefoil is \((0, 3, 4, 1)\). It would seem that crossing a critical value corresponding to a critical point of index 1 corresponds to an increase in genus by 1 for each critical point, whilst crossing a critical point of index 2 corresponds to a decrease in genus by 1 for each critical point.
This relationship between the indices of critical points and Morse codes was observed for other knot types, including a figure-8 knot and a cinquefoil, whose Morse codes are depicted in Figures 6 and 7.

4. Main theorems on Morse codes

We will now adopt the labels from Section 2 with a fixed knot $K$ and potential $\Phi$. We now come to our theorem, relating the index of critical points to the Morse code.

**Theorem 2.** Suppose $(G_0, \ldots, G_N)$ is the distinct Morse code of $K$.

1. If $p_i$ has index 1, then $G_i = G_{i-1} + 1$.
2. If $p_i$ has index 2, then $G_i = G_{i-1} - 1$.

Observe that since $\Phi$ is harmonic, all critical points have index 1 or 2, save for the point at infinity which has index 0.

**Proof.** We start with the first statement. Suppose $p_i$ has index 1. Let $M_i = \Phi^{-1}(0, \Phi(p_i) + \epsilon)$, which is a handlebody submanifold of $S^3$, and let $S_i = \partial M_i$, which as we saw, is a smooth, compact, orientable surface of genus $G_i$. By the Morse reconstruction lemma, $M_i$ is homotopy equivalent to $M_{i-1}$ with a one dimensional disc attached. Hence, by taking a tubular neighborhood in $S^3$, we can see it is homotopy equivalent to a handlebody homeomorphic to $M_{i-1}$ with another handle attached. Therefore, $G_i = G_{i-1} + 1$. 

The second statement involves a somewhat roundabout argument. Suppose $p_i$ has index 2. Then $M_i$ is homotopy equivalent to $M_{i-1}$ with a two dimensional disc attached. Let $M = \Phi^{-1}([0, V_i - \varepsilon])$, where $\varepsilon$ is sufficiently small such that $M$ is homeomorphic to $M_{i-1}$, which in turn implies $\partial M = S = \Phi^{-1}(V_i - \varepsilon)$ is homeomorphic to $S_{i-1}$. Because $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N$, with the inclusions being strict by distinctness of the critical set, this disc is attached via an immersion $\gamma : S^1 \to S$, and we can say $\gamma \in \pi_1(S)$.

The fundamental group $\pi_1(S)$ is the free group on $2G_{i-1}$ generators modulo a product of $G_{i-1}$ commutators of paired generators. Observe that $\pi_1(S_i)$ is the quotient of $\pi_1(S)$ by a certain word $w$ which represents $\gamma$. Attaching a disc to a topological space quotients the original fundamental group by the attaching map, because the path around the disc is nullhomotopic in the new space. Hence, by the classification of compact orientable surfaces by their fundamental groups, $G_i \leq G_{i-1}$, since $\pi_1(S_i)$ does not have more generators than $\pi_1(S)$. We show this inequality is strict by proving $w$ is not the trivial word.

**Figure 8.** The attachment of a 2-disc to $S_{i-1}$ (not necessarily of genus 2) via a nullhomotopic loop. Note the attachment actually occurs in the interior of the solid surface, as the gradient points inwards in the equipotential surfaces.

Suppose $w$ is trivial. Then $\gamma$ bounds a disc on $S_{i-1}$. Note that the Morse Reconstruction Theorem states $\gamma$ does not self-intersect. By the construction of cell complexes, we also have that $\gamma$ bounds another embedded disc in $\mathbb{R}^3 - S_{i-1}$. See the above figure. The union of these two discs forms an embedded 2-sphere $S$. By the Jordan-Brouwer separation theorem, we have that $S_{i-1} \cup S$ partitions $S^3$ into three disjoint connected components.

By taking a tubular neighborhood and then taking the boundary, we can see that $S_i$ is a disjoint union of $S_{i-1}$ and a sphere $S^2$. The flux integral of the electric field over a connected component of an equipotential surface is strictly positive. Therefore, each of these equipotential surfaces encapsulates electric charge by Gauss’ Law, but this contradicts the fact that our charge distribution, the knot $K$, is connected by assumption. Hence, $w$ is nontrivial and we can see that $G_i < G_{i-1}$.

Now we can use Lemmas 1 and 2 to show $G_i = G_{i-1} - 1$. There are $m_1$ instances of consecutive terms in the Morse code increasing by 1 by the first statement of the theorem. Therefore, there are $m_2 = m_1 - 1$ instances of consecutive terms of the Morse code decreasing by a strictly positive integer. Since the Morse code begins at 0 and ends at 1, we can deduce that each of these reductions must be a reduction by 1. □

We now state a corollary on the structure of a Morse code.

**Corollary 3.** The lowest nonzero critical value $V_1$ of a distinct Morse code cannot correspond to a critical point of index 2. Therefore, $G_1 = 1$.

**Proof.** If $V_1$ corresponds to a critical point of index 2, we would have that $\Phi^{-1}([0, V_1 + \varepsilon])$ is homotopic to a 2—sphere with a 2—disc attached. But as we saw in the proof of Theorem 1, we would get a contradiction because the attaching map would be nullhomotopic, as $S^2$ is simply connected. □
Remark: The proof of the main result from [6] uses the Morse Rearrangement Lemma, which shuffles around the critical values whilst leaving the critical set and its index data unchanged. This means one can replace \( \Phi \) with another Morse function whose first nonzero critical value corresponds to a critical point of index 2. However, this does not contradict the preceding corollary because the replacement is not necessarily harmonic. Hence, the flux integrals of the electric field over the replacement’s level surfaces are not necessarily strictly positive.

As stated earlier, many knot embeddings do not have distinct Morse codes. However, there is a straightforward generalization of Theorem 1, whose proof we omit. This theorem relies on the fact that critical points of harmonic functions are isolated, applying the method of proof from Theorem 1 simultaneously across all critical points of the same potential.

**Theorem 4.** Now assume \( K \) does not necessarily have a distinct Morse code. Suppose \( p_1, \ldots, p_m, q_1, \ldots, q_n \) be all of the critical points of \( \Phi \) with the same critical value \( V \), such that each \( p_i \) has index 1 and each \( q_j \) has index 2. If the preceding equipotential surface has genus \( G \), then the successive equipotential surface has genus \( G + m - n \). Therefore, the Morse code is of the form \((0, \ldots, G, G + m - n, \ldots, 1)\). We also have that when \( V = V_1 \), \( m > n \).

5. A Generic Classification of Potential Bifurcations Due to Knot Isotopy

We now consider the effects of knot isotopy on the electric potential. Up until now, we have assumed the electric potential is Morse, which is a generic property in the space of real-valued functions on the knot complement. A knot isotopy induces a path in this space via the corresponding potentials, but the path could pass through non-Morse functions. The proofs of our theorems make liberal use of the Morse property, and break down when \( \Phi \) is not Morse.

However, using Cerf theory, we can extract a generic property of knot isotopies in which the path through non-Morse potentials occurs only finitely many times, and their degenerate critical points can be classified based on partial derivatives of the isotopy. We will see that these degenerate critical points correspond to saddle-node bifurcations. It should be emphasized that this result does not describe the dynamics of a moving charged knot, as we do not take into account the induced magnetic forces from the moving charges. Rather, we are describing a bifurcation among a smooth family of possible fixed, rigid knot configurations.

We now state the Cerf structure theorem. There is a more general structure theorem regarding the coordinate expansion of general degenerate critical points, but we will state the theorem in the context of a homotopy of Morse functions. A full exposition can be found in Ch. 23 of [1].

**Theorem 5.** Let \( \tilde{F} : M \times [0, 1] \to \mathbb{R} \) be a one parameter family of smooth functions on an \( n \)-manifold \( M \) indexed by the variable \( s \). We can replace \( \tilde{F} \) with a substitute function (which we still call \( \tilde{F} \)) in a dense subspace of \( C^\infty(M \times [0, 1]) \) which satisfies the following. Each \( F_s = \tilde{F}(-, s) \) is Morse except for finitely many \( s \) where there could exist degenerate critical points. For each such \( s_0 \), and for each degenerate critical point \( p \) of \( F_{s_0} \), there exist local coordinates \((x_1, \ldots, x_n, s)\) of \( M \times [0, 1] \) centered at \((p, s_0)\) such that \( \tilde{F}(x_1, \ldots, x_n, s) = x_1^3 + \varepsilon_1 x_1 + \varepsilon_2 x_2^3 + \cdots + \varepsilon_n x_n^2 + C \), where each \( \varepsilon_i \in \{\pm 1\} \), whose values depend on the partial derivatives of \( \tilde{F} \), and \( C \) is the degenerate critical value.

We will apply this theorem to the one parameter family of electric potentials induced by a knot isotopy. Suppose \( K_2 \) and \( K_1 \) are two smooth isotopic knots where the isotopy is parametrized by \( r : [0, 2\pi] \times [0, 1] \to \mathbb{R}^3 \). For fixed \( s_0 \in [0, 1] \), we will say \( r_{s_0} = r(-, s_0) \) is the parametrization of the knot \( K_{s_0} \), whose electric potential is \( \Phi_{s_0} \). We will now apply Theorem 4 to the one parameter family \( \Phi_s \), and if necessary, replace \( \Phi_s \) with another nearby generic family so that the property mentioned in the theorem holds.
We will label the coordinates from Theorem 4 as \((x, y, z, s)\) so we have that the local coordinate formula of \(\Phi\) at a specific \(s_0\) where \(\Phi_{s_0}\) has a degenerate critical point is \(\Phi(x, y, z, s) = x^3 + \varepsilon_1 sx + \varepsilon_2 y^2 + \varepsilon_3 z^2 + C\). The convention for a knot isotopy is to use a globally defined \(s\) ranging from 0 to 1, but in our current coordinate chart, the local \(s\)-coordinate is centered at zero, and can be positive or negative.

We can classify degenerate critical points of the electric potential as bifurcations according to the signs of \(\varepsilon_1, \varepsilon_2, \) and \(\varepsilon_3\). In these coordinates, the gradient and Hessian (with respect to the space coordinates alone) are

\[
\nabla \Phi_s(x, y, z) = (3x^2 + \varepsilon_1 sx, 2\varepsilon_2 y, 2\varepsilon_3 z)
\]

\[
H \Phi_s(x, y, z) = \begin{bmatrix}
6x & 0 & 0 \\
0 & 2\varepsilon_2 & 0 \\
0 & 0 & 2\varepsilon_3
\end{bmatrix}.
\]

For fixed \(s \neq 0\), \(\Phi_s\) must be Morse, and within this chart, \(\Phi_s\) has critical points at \(x = \pm \sqrt{-\frac{x^1 s}{3}}, y = 0, z = 0\), which means there are either two critical points when \(\varepsilon_1\) and \(s\) differ in sign, or no critical points when \(\varepsilon_1\) and \(s\) are of the same sign. Furthermore, for a fixed \(s\) where there are two critical points, they are of differing indices of 1 and 2, as the two \(x\)-values of the critical points differ in sign. We call these two critical points a critical pair.

Not every possible configuration of signs for the \(\varepsilon_i\) will result from a potential homotopy induced from a knot isotopy, as each \(\Phi_s\) is harmonic and hence cannot contain local maxima or minima. For instance, when \(\varepsilon_2 = \varepsilon_3 = -1\), the Hessian formula implies the existence of a critical point of index 3 when \(\text{sgn}(s) = -\text{sgn}(\varepsilon_1)\), which cannot exist. This classification is summarized in Figure 9.

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\
1 & 1 & 1 & \text{Impossible} \\
1 & 1 & -1 & \text{Destruction of critical pair} \\
1 & -1 & 1 & \text{Destruction of critical pair} \\
1 & -1 & -1 & \text{Impossible} \\
-1 & 1 & 1 & \text{Impossible} \\
-1 & 1 & -1 & \text{Creation of critical pair} \\
-1 & -1 & 1 & \text{Creation of critical pair} \\
-1 & -1 & -1 & \text{Impossible}
\end{array}
\]

Figure 9. The generic classification of bifurcations of critical points of the potential induced by knot isotopy.

6. Concluding Remarks

With the aid of computer simulations to formulate conjectures, we have described a general relationship between critical sets of electric potentials for charged knots, and the topology of their equipotential surfaces. We then made a novel application of Cerf theory to generically classify the bifurcation of these critical points, demonstrating they occur in pairs.

There are still unresolved questions in electromagnetic knot theory. All of the results in this article apply for general knots, but the nature of Morse codes and critical sets for more specific types of knots, such as hyperbolic or torus knots, remains to be fully examined. Furthermore, even though distinctness of the critical set is a generic property of knots, many knot parametrizations are of interest to knot theorists.
precisely because of their nongeneric symmetry. The counting argument in the last example of Section 2 could be applied to other symmetric knots, possibly yielding stronger conclusions than the results of this article.

There are also issues with computational expense of the numerical simulations, as the computation of critical sets and equipotential surfaces for complex knots requires significant amounts of processing power. The author was able to reduce computation time by significant margins by implementing multicore parallel processing to divide the multivariable Newton method into smaller steps. However, there are further optimizations that have yet to be implemented, such as the use of the Fast Multipole Method in the evaluation of the potential integrals. Should these issues be resolved, we could produce experimental data for a larger class of knots, and formulate conjectures for the unresolved questions in electromagnetic knot theory.

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