ALICE N. NADEAU | RESEARCH STATEMENT

My work is motivated by modeling the climate of Earth and other planets using conceptual climate models. Conceptual models identify predominant mechanisms affecting planetary climate and give rise to a host of mathematical questions. These models have also contributed to scientists' understanding of a wide range of climate phenomena on Earth (such as how increased greenhouse gas emissions will affect global temperature) and other planets (such as the stability of Pluto's nitrogen glaciers). Analyzing conceptual climate models requires techniques from the fields of smooth and nonsmooth dynamical systems, celestial mechanics, harmonic analysis, and data analysis. The results of my work will advance mathematical knowledge of these topics and provide tools for scientists to validate hypotheses about planetary climate.

My long-term research goals include developing dynamical systems techniques that support climate modeling applications as well as addressing mathematical questions that arise in these applications. I address these objectives from three different angles:

(1) placing conceptual climate models and their components on firm mathematical footing,

(2) incorporating data into conceptual modeling frameworks, and

(3) analyzing general dynamical systems which may have applications to science at large.

My recent results in these areas fall mostly under the umbrella of the first and second categories while current work addresses issues in the third category. Plans for future work include projects in all three categories as well as developing necessary bridges between categories.

Towards (1), I use techniques from harmonic analysis to simplify computations of the sunlight distribution across a planet's surface [21, 22]. Because sunlight distribution is a necessary component of climate models of all complexities, these results allow modelers to optimize computations in a wide range of climate models. Additionally, in [24] I show that stable, asymmetric solutions are possible in a conceptual climate model even in the absence of asymmetric forcing. When applied to recent data from Pluto, the results of [24] suggest that the distribution of volatile ices on Pluto is primarily due to annual average sunlight. In the direction of (2), I worked with two ecologists at the University of Minnesota to develop a new framework of data analysis to understand changes in atmospheric carbon [23]. The framework is designed to be accessible and useful to non-mathematicians. Towards (3), I, along with collaborators at Mt. Holyoke College, Sandia National Labs, and Cerner Corp., devised a new method to detect rate-dependent critical transitions in one and two dimensional systems [16]. This is a first step to understanding similar transitions that have been observed in more complicated ecological systems.

My future work both builds on my completed work as well as exploring important new directions. This work includes:

- understanding approximations of sunlight distribution on slowly rotating planets with high obliquity and eccentricity (extension of [21, 22])
- adapting conceptual models to planets with rotation-revolution resonances (e.g. Mercury)
- developing a mathematical foundation for discontinuity boundaries in nonsmooth systems as they appear in some conceptual models (motivated by questions in [24])
- understanding the role of oxygen and the biosphere in the carbon cycle (extension of [23])
- developing mathematical tools to understand rate-dependent critical transitions in scaled systems (extension of [16])

In the following sections I provide details on selected past results and planned next steps.

1 MATHEMATICAL CHALLENGES IN CONCEPTUAL MODELS

Conceptual climate models have contributed to scientists' understanding of a wide range of climate phenomena, including the possibility of a completely glaciated Earth [1, 35, 38] and how long we can expect the ice caps to last [5, 33]. Recently, scientists have turned to conceptual models to study climate phenomena on other planets, including the stability of Pluto's nitrogen glaciers [11, 12] and the possibility of liquid water on planets orbiting the nearby star TRAPPIST-1 [6, 14]. My work addresses mathematical challenges raised by adapting these models to other planets.

1.1 EFFECTS OF PLANETARY ROTATION RATE ON SUNLIGHT DISTRIBUTION

An important component in climate models of all complexities is insolation (from *in*coming *sol*ar radi*ation*). The annual mean insolation distribution for a rapidly rotating planet (e.g. Earth) has a closed form as a result of an application of Birkhoff's ergodic theorem allowing one to interchange integration over a year to integration over longitude. The distribution as a function of the latitude, λ, and the obliquity (axial tilt), β, is given by the integral equation

$$s(\lambda, \beta) = \frac{2}{\pi^2} \int_0^{2\pi} \sqrt{1 - (\cos\lambda \sin\beta \sin\gamma - \sin\lambda \cos\beta)^2} \, d\gamma \tag{1}$$

where γ denotes longitude [10, 19]. On the other hand, researchers at NASA have conducted a handful of numerical studies to understand insolation on slowly rotating planets with small integer ratios between their rotation and revolution rates (e.g. Mercury with 3 rotations per 2 revolutions) [9, 10], finding that these resonant cases have large longitudinal differences in their distributions. Critically, it is not known where the cut-off between "rapid" and "slow" rotation lies. This cut-off is crucial in modeling climate because insolation on rapidly rotating planets is symmetric about the axis of rotation, allowing highly accurate polynomial approximations. Conversely, calculating insolation on planets with slow, resonant rotation requires numerical integration over an entire revolution. Furthermore, it is not known if it is best to approximate slow, non-resonant rotation with the rapid rotation formula, the slow rotation method, or a new approach entirely.

In [21] I showed that it is possible to approximate the average annual insolation for any rapidly rotating planet to a high degree of accuracy with the sixth degree Legendre polynomial approximation of (1). This polynomial approximation is needed because the integral in (1) cannot be computed explicitly. In [22] I generalized the results of [21], showing that, surprisingly, the full Legendre series expansion of (1) is a symmetric function. In particular:

Theorem (Nadeau and McGehee, [22]). *The annual average insolation distribution function can be written*

$$s(\lambda, \beta) = \sum_{n=0}^{\infty} A_{2n} P_{2n}(\cos\beta) P_{2n}(\sin\lambda),$$

where P_{2n} is the Legendre polynomial of degree 2n, and where

$$A_{2n} = \frac{(-1)^n (4n+1)}{2^{2n-1}} \sum_{k=0}^{n} \binom{2n}{n-k} \binom{2n+2k}{2k} \binom{1/2}{k+1}.$$

This result allows modelers to choose the approximation that gives them their desired degree of accuracy by truncating the series at arbitrary n. Error bounds in n are provided in my thesis [25]. The proof applies techniques from complex analysis and the addition theorem for spherical harmonics.
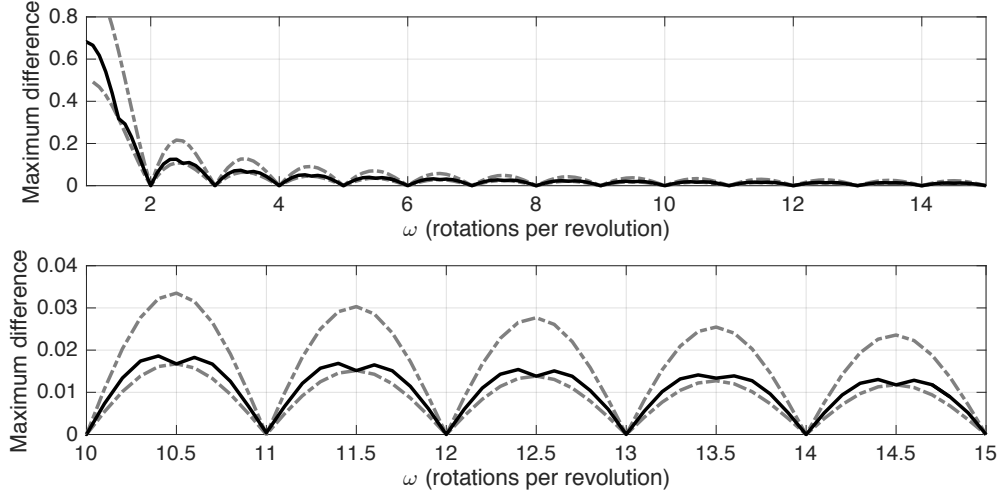
Figure 1: *Top: Plot of the norm (solid black curve) and the upper and lower bounds from* (2) *(dash-dot gray curves). Bottom: The tail of the top figure plotted with an expanded vertical scale.*

In my thesis, I focus on finding the cut-off point for what constitutes "rapid" rotation [25]. I bound the maximum difference by longitude between the actual annual insolation distribution and the calculation one would get assuming rapid rotation, showing that this difference decays like the reciprocal of the rotation rate. For circular orbits this decay can be explicitly computed:

Theorem (Nadeau [25]). *For a planet on a circular orbit with zero obliquity, if $\tilde{I}(\gamma, \lambda, \omega)$ represents the annual average insolation as a function of longitude γ, latitude λ, and rotation rate ω; $\hat{I}(\lambda, \omega) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{I}(\gamma, \lambda, \omega) d\gamma$; and $\omega > 1$, then*

$$\frac{|\sin(\pi\omega)|}{2\pi(\omega - 1)} \le \sup_{\phi \in [0, 2\pi]} \|\tilde{I}(\gamma, \lambda, \omega) - \hat{I}(\lambda, \omega)\|_{\mathcal{C}^0} \le \frac{|\sin(\pi\omega)|}{\pi(\omega - 1)}. \tag{2}$$

Furthermore, equality is achieved in the lower bound when 2ω is an integer (see Figure 1).

1.2 ASYMMETRIES IN ENERGY BALANCE MODELS

The Budyko–Widiasih latitude-dependent energy balance model (EBM) has been widely used to understand Earth's contemporary climate [5, 36], Earth's glacial cycles [1, 4], and the climate of other planets [6, 28]. The Budyko–Widiaish model is based on an EBM proposed by Budyko [5], to which Widiasih appended a dynamic ice line (the latitude north of which there is continually ice) [36]. It has the form

$$\frac{\partial}{\partial t} T(t, y) = F(y, \eta, T), \qquad \frac{d\eta}{dt} = \rho(T(t, \eta) - T_c), \tag{3}$$

and describes the evolution of the temperature profile (T) and the ice line (η) dependent on latitude (y) and average temperature where ice is present year round (T_c). The study of this and other latitude-dependent energy balance models has historically been restricted to the northern hemisphere ([26] and references therein), a constraint that continues to this day [4, 28]. This restriction forces solutions to be symmetric across the planet's equator. However, recent data show that this assumption may be too restrictive when applied to Earth's past climate or other planets [29].
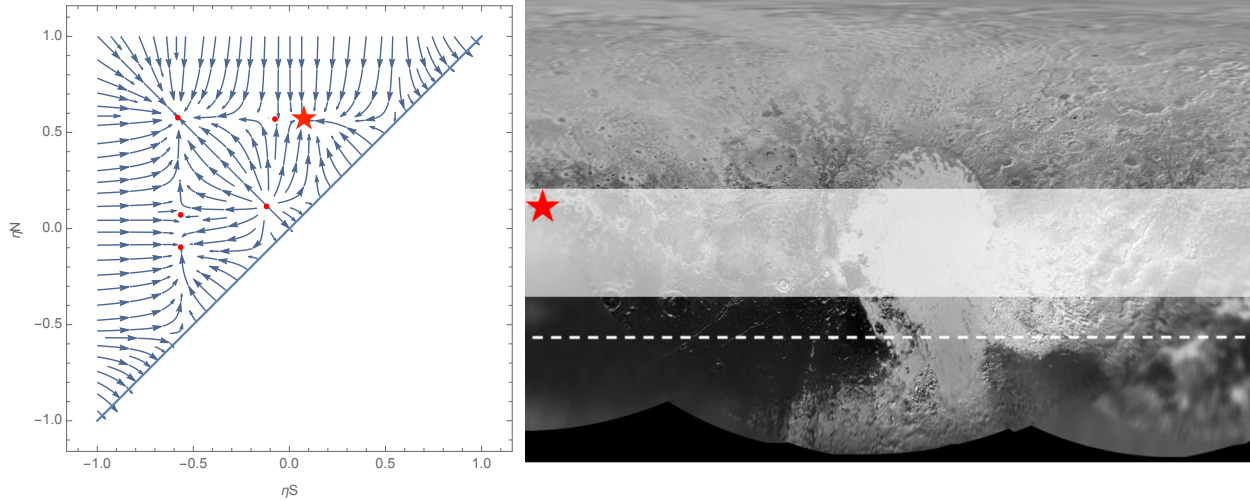
Figure 2: *Left: phase diagram of northern and southern edges of an ice belt on Pluto (from [24]). Red star indicates a stable asymmetric ice belt. Right: The stable asymmetric ice belt (star in lefthand figure) overlaid on a NASA image of Pluto's surface. The dashed line is Pluto's equator.*

In [24], I removed the long-standing symmetry assumptions in an approximation to (3) by amending an additional ice line equation and approximating F using methods analogous to [20]. In this framework the extended system reduces to a three dimensional system of ODEs. I showed that the restriction of (3) to the northern hemisphere works for the range of parameters that define Earth's current climate because symmetric solutions in this case lie on a two-dimensional manifold which is locally attracting in the region of state space which is physically accessible. I also showed that restricting to the northern hemisphere does not capture all possible cases because there are regions of parameter space where the manifold of symmetric solutions is locally repelling and stable, asymmetric ice line configurations exist for (3) [24]. These asymmetric solutions coincide with the current location of ices on Pluto (Figure 2) using parameters informed by data.

1.3 CURRENT AND FUTURE WORK IN CONCEPTUAL MODELS

My current work in conceptual climate models is two-fold. First, I am exploring the dependence of the insolation distribution on other orbital parameters such as obliquity and eccentricity. Preliminary results show that high obliquities reduce the longitudinal differences in the distribution while high eccentricities increase the differences. Second, I am working to adapt Budyko-Widiasih model to slowly rotating planets, building off my previous work and a recent case study for tidally locked planets [6]. In the future I will address the open question: what happens when two ice lines collide? The collision of the ice lines corresponds to a discontinuity boundary (a manifold along which two different vector fields meet) at the boundary of the domain of state space. This contrasts with discontinuity boundaries in classical Filippov systems, which occur only on the interior of state space [13]. To analyze the system within Filippov's framework, one must extend its domain beyond the physically relevant region of state space [13, 36]. Novel methods that avoid this nonphysical extension have also been proposed (e.g. [4, 20]), but it is not clear to what larger class of systems these methods might apply. I plan to build on these results and my own work to address this issue in generic nonsmooth systems.
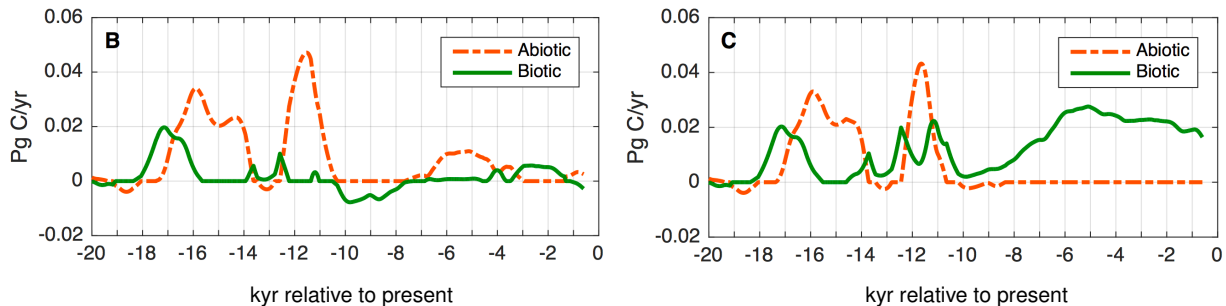
Figure 3: *Left: Contributions from biotic (solid, green) and abiotic (dot-dashed, orange) carbon pools affecting changes in atmospheric carbon. Positive values indicate a sources of carbon, negative values a sink Right: Same as lefthand figure but after accounting of a known large postglacial biotic sink. Note the now needed source of biotic carbon. Figures from [23].*

2 DATA ANALYSIS IN CONCEPTUAL CLIMATE MODELS

Understanding past carbon sources and sinks informs predictions of how human activity affects these pools in the present and future. Different physical, biological, and chemical processes discriminate in favor of one carbon isotope over the other, making it possible to identify sources and sinks based on their isotopic ratio. My work to constrain the flow of carbon between various pools in a conceptual carbon cycle model leverages recent carbon isotope time series.

2.1 A NEW DATA ANALYSIS METHOD FOR CARBON ISOTOPES

A recent study in *Science* measured the abundance of carbon isotopes in the atmosphere for the last 25,000 years [32]. Based on the isotopic ratio in the atmosphere, Schmitt et al. identified possible times when the biosphere or the oceans were having the largest effect on the amount of carbon in the atmosphere. They concluded that between 12,000 and 7,000 years ago there was a net effect of sequestering carbon from the atmosphere into the biosphere and that the amount of carbon in the atmosphere equilibrated from 7,000 years ago to the industrial revolution [32].

In [23] I worked with UMN ecologists to develop a conceptual box model of the carbon cycle. Our analysis finds the minimal amount of carbon exchanged between boxes in each time step in order to match the data in Schmitt et al. For example, in the simplest case, if the observed change has a biotic isotopic signature, then the algorithm attributes all changes in that time step to biotic pools. In more complicated cases where the observed change has an isotopic signature that falls outside of the biotic or abiotic ranges, the algorithm determines the contribution from these pools which minimizes the net flow. Our work confirms Schmitt et al.'s analysis [32] and scientific hypotheses on anthropogenic emissions from early agricultural practices [30]:

Main Result (Nadeau, Lehman, McGehee, Gorham [23]). *Over the past 16,000 years decreases in atmospheric carbon corresponded to sequestration into biological pools and increases corresponded to abiotic releases of carbon, e.g. from the oceans or volcanos. Furthermore, including a known large post-glacial biotic sink into the analysis yields the presence of a biotic source starting about 11,000 years ago (see Figure 3 for plots of results).*

The results of the analysis are unique up to the sign of the net contributions from each carbon pool, i.e. sources are always sources and sinks are always sinks. Furthermore, we show that our results are robust between different data sets over the same time period [23].

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2.2 CURRENT AND FUTURE WORK ON THE CARBON BUDGET

We plan to use our algorithm to study contemporary sources and sinks of carbon, especially with regards to the role the biosphere plays in mitigating human greenhouse gas emissions. Preliminary results indicate a larger biological carbon sink than other studies in the field and are currently being prepared for publication. Furthermore, I continue to work on this project with Minnesota ecologists to understand the role that oxygen plays in the carbon cycle and whether incorporating oxygen data into our analysis changes the qualitative behavior of the carbon-only results. In particular, we are curious if incorporating oxygen data changes a source to a sink or vice versa.

3 CRITICAL TRANSITIONS IN NONAUTONOMOUS SYSTEMS

Tipping points in the scientific literature are characterized by a sudden, qualitative shift in the behavior or state of the system due to a relatively small change in inputs [18, 31]. In mathematical models, rapid shifts in parameters can cause tipping phenomena even in the absence of bifurcations or noise [2]. However, this rate-induced tipping phenomenon is not yet well-defined; there is no definition that encompasses all examples within the scientific literature. My work in this area sets out criteria to detect tipping in idealized ecological models. Systems of the form I study are not covered by the framework of other tipping studies. This work is a first step to understanding rate-dependent critical transitions in more complicated climate models and the real climate system.

3.1 DETECTING RATE-DEPENDENT TRANSITIONS WITH STABILITY SPECTRA

Recent studies have set out criteria to detect tipping from various realms of well-established mathematical theory [2, 7, 37]. While there is a solid foundation for one dimensional equations [2, 3, 27], limited theoretical work has been conducted to understand higher dimensional systems, especially when the tipping is caused by scaling in the system. In particular, tipping has almost exclusively been studied in systems of the form $\dot{x} = f(x - \lambda(at))$ where $x \in \mathbb{R}$ or $\mathbb{R}^2$ [2, 27]. Recent advances characterize tipping in the fully general one dimensional equation $\dot{x} = f(x, \lambda(at))$ under the assumption that $\lambda(at)$ is bi-asymptotically constant [15].

My collaborators and I investigate the phenomenon of rate-dependent tipping through the framework of nonautonomous bifurcations. In [16] we analyze the stability of a trajectory in a nonautonomous system using Lyapunov and Steklov spectra and their numerical approximations (given in [8]). We show that the numerical approximations of these spectra can be used to find the short term growth and decay rates of a trajectory. Because rate-dependent tipping involves finding the critical rate at which there is a loss of stability locally in time, these spectra can determine whether or not a trajectory tips in some cases. In particular, we find that we can *predict* tipping in a handful of cases before the solution crosses a point of no return (see Figure 4).

3.2 CURRENT AND FUTURE WORK ON RATE-DEPENDENT TRANSITIONS

I am working with my collaborators to study rate-dependent transitions in systems of the form

$$\begin{aligned}
\dot{x} &= \lambda(at)f_1(x,y) + f_2(x,y) \\
\dot{y} &= g(x,y)
\end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\lambda(t)$ is as smooth as the system (at least $\mathcal{C}^1$) and a is the rate parameter controlling the speed of change in λ. Tipping in systems of the form (4) has been documented in
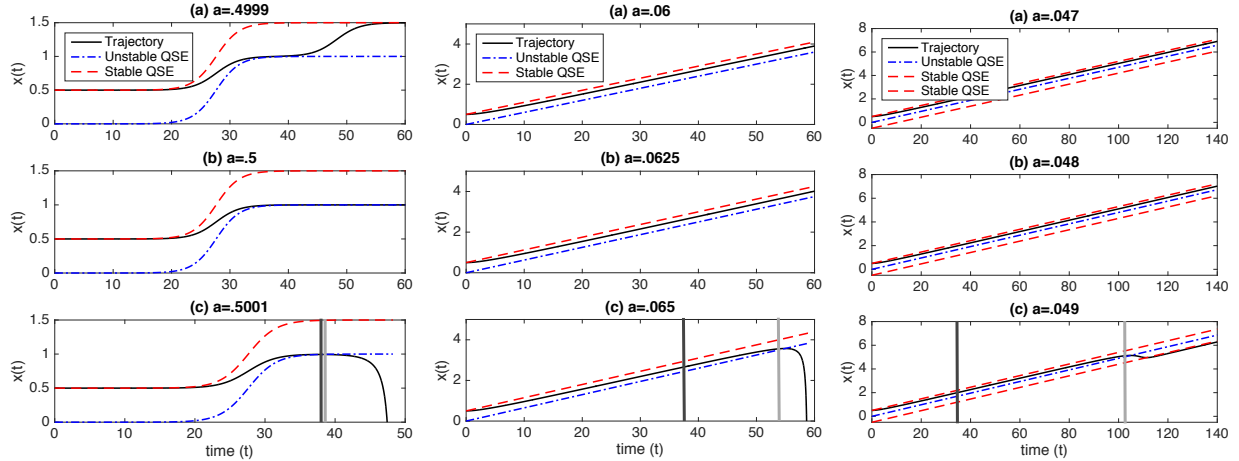
Figure 4: *Rate-induced tipping in three systems due to the rate "a" indicated above each figure. Trajectories in (a) and (b) do not tip, even though they move away from the stable quasi-static equilibrium (QSE) for some time. Trajectories in (c) tip. Previous methods are able to detect tipping at the vertical gray bar, our method is able to detect tipping at the vertical black bar.*

ecological and climate models [2, 34, 37]. For example, in a conceptual model of surface temperature and greenhouse gases released from bogs, releasing carbon at a slow, linear rate ($\lambda'(at) = a$) causes a slow increase in global temperature but at a fast enough rate causes a large emission of carbon into the atmosphere, raising global temperatures several degrees in less than a decade [2, 37]. Our preliminary investigations on this project show that tipping can occur in systems of the form (4) even when $\lambda(at)$ is not asymptotically constant [16]. However, there is no theoretical framework to understand this behavior. We will build on our preliminary results to fill this gap.

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