Math 1272 Quiz 8 Name: Solutions

Time allowed: 20 minutes

1. (30 points) Use series to evaluate

$$\lim_{x\to 0}\frac{1+x-e^x}{\cos(x)-1}.$$

$$1+x-e^x = 1+x-\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = \cancel{1}+\cancel{x}-\left(\cancel{1}+\cancel{x}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)$$

$$= -\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right) = -x^2\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right)$$

$$\cos x - 1 = \left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right)-1 = \left(\cancel{1}-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)-\cancel{1}$$

$$= \left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right) = -x^2\left(\frac{1}{2!}+\frac{x^2}{4!}+\cdots\right)$$

$$\lim_{x\to 0}\frac{1+x-e^x}{\cos x-1} = \lim_{x\to 0}\frac{\cancel{-x^2}\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right)}{\cancel{-x^2}\left(\frac{1}{2!}+\frac{x^2}{4!}+\frac{x^4}{4!}+\cdots\right)}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{2!}+\frac{\cancel{x}}{3!}+\frac{x^2}{4!}+\cdots\right)}{\left(\frac{1}{2!}+\frac{x^2}{4!}+\frac{x^4}{4!}+\cdots\right)} = \frac{\frac{1}{2!}}{\frac{1}{2!}} = \boxed{1}$$

2. (a) (*20 points*) Fully state Taylor's Inequality. Be sure to include all necessary hypotheses.

If $|f^{(n+1)}(x)| \le M$ for $|x-a|\le d$ then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

for $|x-a|\le d$

(b) (10 points) Suppose that $f(x)$ has a Maclaurin series that converges on $[-1,1]$ and $|f^{(n)}(x)| \le 10$ at all $|x|\le 1$ and all n. Which of the following is a degree of a Taylor polynomial for f which will allow us to approximate $f(0.1)$ if we want the error to be less than $5\cdot 10^{-6}$? (Hint: Use Taylor's Inequality from part (a))

(i) 2

(ii) 3

(iii) 4

(iv) 5

$$|R_n(.1)| \le \frac{10(.1)^{n+1}}{(n+1)!} = \frac{10}{(n+1)!\,10^{n+1}}$$

$$R_2(.1) \le \frac{10}{3!\,10^3} = \frac{1}{3!\,10^2} > 5\cdot 10^{-6}$$

$$R_3(.1) \le \frac{10}{4!\,10^4} = \frac{1}{4!\,10^3} = \frac{1}{24\cdot 10^3} \approx \frac{5}{10^2\cdot 10^3} > 5\cdot 10^{-6}$$

PLEASE SEE THE OTHER SIDE FOR MORE PROBLEMS.

$$R_4(.1) \le \frac{10}{5!\,10^5} = \frac{1}{120\cdot 10^5} \le \frac{1}{10^7} < 5\cdot 10^{-6}$$

$$T_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)(x-3)^2}{2!} + \frac{f'''(3)\,(x-3)^3}{3!}$$

3. Let $f(x)$ denote the value of the series $\sum_{n=2}^{\infty} 2^n(x-3)^n/n$ at points where it converges.

   (a) (15 points) What is the 3rd degree Taylor polynomial of $f(x)$ at 3?

$$f(x) = \sum_{n=2}^{\infty} \frac{2^n (x-3)^n}{n}, \quad f(3) = 0$$

$$f'(x) = \sum_{n=2}^{\infty} 2^n (x-3)^{n-1}, \quad f'(3) = 0$$

$$f''(x) = \sum_{n=2}^{\infty} (n-1)\, 2^n (x-3)^{n-2}, \quad f''(3) = 4$$

$$f'''(x) = \sum_{n=2}^{\infty} (n-2)(n-1)\, 2^n (x-3)^{n-3}, \quad f'''(3) = 16$$

$$T_3 = 0 + 0(x-3) + \frac{4(x-3)^2}{2} + \frac{16(x-3)^3}{6} = \boxed{2(x-3)^2 + \frac{8(x-3)^3}{3}}$$

   (b) (15 points) What is the radius of convergence of $f(x)$?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{n+1} \cdot \frac{n}{2^n (x-3)^n} \right|$$

$$= \lim_{n\to\infty} \frac{2n}{n+1} |x-3| \Rightarrow 2|x-3| < 1$$

$$|x-3| < 1/2$$

Radius of Convergence is 1/2

   (c) (10 points) What is the radius of convergence of the Taylor series of $f'(x)$ at 3? Give your reasoning.

The radius of convergence is still 1/2 because taking the derivative doesn't change the radius of convergence of a power series.

PLEASE SEE THE OTHER SIDE FOR MORE PROBLEMS.