POLISH MODULES OVER SUBRINGS OF $\mathbb{Q}$

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Abstract. We give a method of producing a Polish module over an arbitrary subring of $\mathbb{Q}$ from an ideal of subsets of $\mathbb{N}$ and a sequence in $\mathbb{N}$. The method allows us to construct two Polish $\mathbb{Q}$-vector spaces, U and V, such that
— both U and V embed into $\mathbb{R}$ but
— U does not embed into V and V does not embed into U,
where by an embedding we understand a continuous $\mathbb{Q}$-linear injection. This construction answers a question of Frisch and Shinko [4]. In fact, our method produces a large number of incomparable with respect to embeddings Polish $\mathbb{Q}$-vector spaces.

1. Introduction

1.1. **The background for and outline of the main results.** This is a paper on a connection between Descriptive Set Theory and Commutative Algebra. In [4], Frisch and Shinko initiated the study of Polish modules and embeddings among them. The present work answers a question from that paper [4, Problem 4.4] by producing incomparable with respect to embeddings Polish R-modules for subrings R of $\mathbb{Q}$. In fact, we give a rather general method of constructing Polish R-modules for such R. This construction associates Polish R-modules with certain ideals of subsets of $\mathbb{N}$ (translation invariant analytic P-ideals) and certain sequences in $\mathbb{N}$ (base sequences) under the assumption of a suitable coherence between the ideals and the sequences. The procedure appears to be quite canonical and may have other applications.

Before explaining the context of the question from [4] and our construction, which will require some definitions, we formulate a statement that is a special case of Corollary 1.2. The statement does not require any extra definitions as it can be phrased entirely in the language of vector spaces over $\mathbb{Q}$ and the Borel structure of $\mathbb{R}$; on the other hand, it is a representative consequence of our main results. We treat the real numbers $\mathbb{R}$ as a $\mathbb{Q}$-vector space.

There exists a family F_x, for $x \subseteq \mathbb{N}$, of uncountable Borel $\mathbb{Q}$-vector subspaces of $\mathbb{R}$ such that, for all $x, y \subseteq \mathbb{N}$,

— *if $x \setminus y$ is finite, then $F_x \subseteq F_y$ and*

— *if $x \setminus y$ is infinite, then each Borel $\mathbb{Q}$-linear map $F_x \to F_y$ is constantly equal to zero.*

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Each F_x, for $x \subseteq \mathbb{N}$, can be taken to be a countable union of closed subsets of $\mathbb{R}$.

Now, we recall the notion of Polish module from [4]. Actually, we recall only the special case of the definition (when the ring is commutative and countable) that is relevant to stating the problem from [4]. Let R be a commutative countable ring (with unity 1). By a **Polish R-module** M we understand an abelian Polish group, where $+$ denotes the group operation, and a continuous function

$$R \times M \ni (r, m) \to r \cdot m \in M,$$

with R equipped with the discrete topology, such that

$$r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2, \;\; (r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m,$$
$$r_1 \cdot (r_2 \cdot m) = (r_1 r_2) \cdot m, \;\; 1 \cdot m = m,$$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. Of course, if R is a countable field, then an R-module is a vector space over R. For the record, we also recall the notion of homomorphism and embedding among R-modules. If M_1 and M_2 are R-modules, a function $f\colon M_1 \to M_2$ is called an **R-module homomorphism** if it is a group homomorphism and $f(rm) = rf(m)$ for all $r \in R$ and $m \in M_1$. An injective R-module homomorphism is called an **R-embedding**. We use the suggestive notation from [4]

$$M_1 \sqsubseteq^R M_2$$

to indicate that there is a continuous R-embedding $M_1 \to M_2$; and also

$$M_1 \sqsubset^R M_2$$

if $M_1 \sqsubseteq^R M_2$ and $M_2 \not\sqsubseteq^R M_1$.

It was proved in [4, Theorem 1.2 and Section 3] that

if R is a countable Noetherian ring, then there is a family of uncountable Polish R-modules M_S, parametrized by nontrivial quotients S of R, such that, for each uncountable Polish R-module M, we have $M_S \sqsubseteq^R M$ for some S.

Recall that a commutative ring is **Noetherian** if each strictly increasing under inclusion sequence of ideals is finite. The modules M_S have an explicit definition, but it will not be relevant here. The family of nontrivial quotients S of R is countable; thus, the above statement asserts that the class of uncountable Polish R-modules has a countable basis under R-embeddings. The situation is even more striking when R is a countable field. A field R has only one non-trivial quotient, namely, R itself, and is obviously Noetherian. So the above family of the M_S-s has only one element M_R. Thus,

if R is a countable field, then there is an uncountable Polish R-vector space M_R such that, for each uncountable Polish R-vector space M, we have $M_R \sqsubseteq^R M$.

So the class of uncountable Polish R-vector spaces has a one-element basis under embedding.

A particularly interesting case is that of $R = \mathbb{Q}$. In this case, there is a classical uncountable Polish $\mathbb{Q}$-vector space, namely, $\mathbb{R}$. The above statement gives $M_{\mathbb{Q}} \sqsubseteq^{\mathbb{Q}}$

$\mathbb{R}$. As it was pointed out in [4], there is no continuous $\mathbb{Q}$-embedding from $\mathbb{R}$ to $M_\mathbb{Q}$, so we have

$$M_\mathbb{Q} \sqsubset^\mathbb{Q} \mathbb{R}.$$

Now, a natural question arises whether there exists a Polish $\mathbb{Q}$-vector space V such that

$$M_\mathbb{Q} \sqsubset^\mathbb{Q} V \sqsubset^\mathbb{Q} \mathbb{R}. \tag{1}$$

This question was asked as [4, Problem 4.4]. We answer it in the affirmative by proving the following theorem.

Theorem 1.1. *Let R be a subring of $\mathbb{Q}$ not equal to $\mathbb{Z}$. There exists a family V_x, for $x \subseteq \mathbb{N}$, of uncountable Polish R-modules such that, for all $x, y \subseteq \mathbb{N}$, we have*

(i) $V_x \sqsubseteq^R \mathbb{R}$, *;*
(ii) if $x \setminus y$ is finite, then $V_x \sqsubseteq^R V_y$;
(iii) if $x \setminus y$ is infinite, then each continuous R-module homomorphism $V_x \to V_y$ is identically equal to zero; in particular, $V_x \not\sqsubseteq^R V_y$.

In fact, there exist continuous R-module embeddings $f_x \colon V_x \to \mathbb{R}$ witnessing (i) so that (ii) is witnessed by a continuous R-module emdeding $g\colon V_x \to V_y$ with $f_x = f_y \circ g$. Further, the image of f_x is a countable union of closed subsets of $\mathbb{R}$.

To see clearly the relationship between Theorem 1.1 and [4, Problem 4.4], take $R = \mathbb{Q}$ and note that if $x \subseteq \mathbb{N}$ is such that both x and $\mathbb{N} \setminus x$ are infinite, then V_x as in Theorem 1.1 fulfills (1). Indeed, by Theorem 1.1(i) and the defining property of $M_\mathbb{Q}$, we have

$$M_\mathbb{Q} \sqsubseteq^\mathbb{Q} V_x \sqsubseteq^\mathbb{Q} \mathbb{R}.$$

Set $y = \mathbb{N} \setminus x$. If $\mathbb{R} \sqsubseteq^\mathbb{Q} V_x$, then, by Theorem 1.1(i) for y and transitivity of $\sqsubseteq^\mathbb{Q}$, we would have $V_y \sqsubseteq^\mathbb{Q} V_x$ contradicting Theorem 1.1(iii). If $V_x \sqsubseteq^\mathbb{Q} M_\mathbb{Q}$, then, by definition of $M_\mathbb{Q}$ and transitivity of $\sqsubseteq^\mathbb{Q}$, we would have $V_x \sqsubseteq^\mathbb{Q} V_y$ again contradicting Theorem 1.1(iii). It follows that (1) holds with $V = V_x$.

Note that one can prove the non-embedding statement $\mathbb{R} \not\sqsubseteq^\mathbb{Q} V_x$ in (1) more directly by noticing that $\mathbb{R}$ is connected, while V_x is totally disconnected, since its image under the embedding into $\mathbb{R}$ is totally disconnected (as a nontrivial subspace of $\mathbb{R}$ over $\mathbb{Q}$). Thus, there is no non-constant continuous function from $\mathbb{R}$ to V_x. It was pointed out to us by Josh Frisch that to get the other non-embedding statement in (1) for V_x as above, one can also argue as follows. By the first part of the proof of Lemma 3.4 below, there is an element of V_x, namely 1, whose $\mathbb{Q}$ multiples are dense in V_x. On the other hand, from the definition of $M_\mathbb{Q}$, as given in [4], one proves that $M_\mathbb{Q}$ does not have an uncountable linear subspace over $\mathbb{Q}$ with this property. This gives $V_x \not\sqsubseteq^\mathbb{Q} M_\mathbb{Q}$.

As pointed out in Section 2.2, subrings of $\mathbb{Q}$ are Noetherian; therefore, the above theorem fits into the context from [4] described above even for R not equal to $\mathbb{Q}$.

We have a corollary of Theorem 1.1 that is phrased in terms of the Borel structure of $\mathbb{R}$ only, and which generalizes the statement at the top of the introduction.

Corollary 1.2. *Let R be a subring of $\mathbb{Q}$ not equal to $\mathbb{Z}$. There exists a family F_x, for $x \subseteq \mathbb{N}$, of uncountable Borel R-submodules of $\mathbb{R}$ such that, for all $x, y \subseteq \mathbb{N}$,*

(i) if $x \setminus y$ is finite, then $F_x \subseteq F_y$ and
(ii) if $x \setminus y$ is infinite, then each Borel R-module homomorphism $F_x \to F_y$ is constantly equal to zero.

Each F_x, for $x \subseteq \mathbb{N}$, can be taken to be a countable union of closed sets.

Note that that the Borel complexity (countable unions of closed sets) in the conclusion of Corollary 1.2 is clearly best possible.

Proof of Corollary 1.2 from Theorem 1.1. Let V_x and f_x for $x \subseteq \mathbb{N}$, be as in Theorem 1.1. Let

$$F_x = f_x(V_x).$$

Clearly F_x is an R-submodule of $\mathbb{R}$. Directly from the properties of f_x in Theorem 1.1, F_x is F_σ and (i) holds. If $x \setminus y$ is infinite and $f\colon F_x \to F_y$ is a Borel R-module homomorphism, then $(f_y)^{-1} \circ f \circ f_x$ is an R-module homomorphism from V_x to V_y, which is Borel since the inverse of a Borel bijection is Borel. Thus, by Pettis Theorem [5, Theorem 9.10], it is continuous, and so f is constantly equal to 0 by Theorem 1.1. □

The paper is structured as follows. Theorem 1.1 is a consequence of a general method of constructing Polish modules over subrings of $\mathbb{Q}$, which we present in Section 2; see especially Theorem 2.13. Then, in Section 3, we prove a result, Theorem 3.2, on strong non-containment, under appropriate conditions, of modules produced using the method from Section 2. Finally, in Section 3.2, we apply the results from Sections 2 and 3 to prove Theorem 1.1.

1.2. **Conventions.** By $\mathbb{N}$ we denote the set of all positive integers; so $0 \notin \mathbb{N}$. The power sets of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, is often regarded as a compact metric space via the canonical identification with $\{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$. Further, $\mathbb{R}_{\geq 0}$ stands for the set of non-negative real numbers. To keep notation lighter, for $r, s \in \mathbb{R}$, set

$$r \ominus s = |r - s|. \tag{2}$$

Finally, a subset of a metric space is called

$$F_\sigma$$

if it is a countable union of closed sets.

2. A class of Polish modules

In this section, we present the main construction of the paper. It associates a Polish module with two inputs: an ideal (of a certain type) of subsets of $\mathbb{N}$ and a base sequence adapted to the ideal. These two main objects in the construction—ideals of subsets of $\mathbb{N}$ and base sequences—are defined in Section 2.1. After some preliminary work in Sections 2.2 and 2.3, we define the Polish modules and prove their basic properties in Section 2.4.

Our construction extends the one in [8, Section 3], where Polishable subgroups of $\mathbb{R}$ were associated with certain ideals of subsets of $\mathbb{N}$.

2.1. **Base sequences and ideals of sets of natural numbers.** A sequence

$$\vec{a} = (a_n)$$

is called a **base sequence** if $a_n \in \mathbb{N}$ and $a_n \geq 2$, for each $n \in \mathbb{N}$. A non-empty family I of subsets of $\mathbb{N}$ is called an **ideal** if it is closed under taking finite unions and subsets. Additionally, in this paper, we will always assume that $\{n\} \in I$ for each $n \in \mathbb{N}$. The two types of objects are related to each other through the following notion. A base sequence $\vec{a}$ is called **adapted to** I if

$$\{n \mid a_n \neq a_{n+1}\} \in I.$$

2.2. **Base sequences.** Base sequences are useful to us for two reasons—each such sequence gives a representation of real numbers and it determines a subring of $\mathbb{Q}$.

2.2.1. *Representations of non-negative reals.* Given a base sequence $\vec{a}$, each $r \in \mathbb{R}_{\geq 0}$ has a unique representation as

$$r = [r] + \sum_{n=1}^{\infty} \frac{r_n}{a_1 \cdots a_n}, \tag{3}$$

where $[r]$ is the integer part of r and r_n is such that $0 \leq r_n < a_n$ and $r_n \neq a_n - 1$ for infinitely many n. We leave checking the above assertion to the reader. Obviously, this is analogous to representing r using its decimal expansion. One can extend the representation (3) to negative reals, but we will not need this extention in what follows. For $r \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{N}$, r_n will stand for the n-th digit of r in the representation of r as in (3).

Fix a base sequence $\vec{a}$.

Lemma 2.1. *Let $r, s \in \mathbb{R}_{\geq 0}$.*

(i) If $0 \leq r < \frac{1}{a_1 a_2 \cdots a_l}$, then $r_i = 0$ for all $i \leq l$.
(ii) If $0 \leq s - r < \frac{1}{a_1 a_2 \cdots a_l}$ and ($r_l \neq a_l - 1$ or $s_l \neq 0$), then $s_i = r_i$ for all $i < l$.

Proof. (i) is immediate. To see point (ii), note that, by (i), $(s-r)_i = 0$ for all $i \leq l$. By a straightforward calculation, this condition, in conjunction with $r_l \neq a_l - 1$ or $s_l \neq 0$, implies that $s_i = r_i$ for all $i < l$. $\square$

The next lemma on the continuity of the digits of a number is essentially a qualitative version of Lemma 2.1(ii).

Lemma 2.2. *Let $\big(r(k)\big)_{k \in \mathbb{N}}$, be a sequence in $\mathbb{R}_{\geq 0}$ converging to $r \in \mathbb{R}_{\geq 0}$. If either one of the following two conditions holds*

(i) r has infinitely many non-zero digits,
(ii) $r(k) \geq r$ for all k,

then, for each $n \in \mathbb{N}$, $r(k)_n = r_n$ for large enough k.

Proof. Under assumption (ii), for each l with $r_l \neq a_l - 1$, Lemma 2.1(ii) implies that, for large enough k, we have $r(k)_i = r_i$ for all $i < l$. Since there are infinitely many such l, the conclusion follows. Now, assume (i). By what we just proved, we can suppose that $r(k) < r$ for all k. Now, for each l with $r_l \neq 0$, Lemma 2.1(ii) implies that, for large enough k, we have $r(k)_i = r_i$ for all $i < l$. Since, by assumption (i), there are infinitely many such l, the conclusion follows. □

For $r \in \mathbb{R}_{\geq 0}$ with the representation (3), we let

$$j_{\vec{a}}(r) = \{n \in \mathbb{N} \mid r_n \neq r_{n+1}\}, \tag{4}$$

so $j_{\vec{a}}\colon \mathbb{R}_{\geq 0} \to 2^{\mathbb{N}}$. Lemma 2.2 has an immediate corollary concering continuity properties of $j_{\vec{a}}$.

Lemma 2.3. *Let $\big(r(k)\big)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}_{\geq 0}$ converging to $r \in \mathbb{R}_{\geq 0}$. If either one of the following two conditions holds*

(i) r has infinitely many non-zero digits,

(ii) $r(k) \geq r$ for all k,

then $j_{\vec{a}}(r(k)) \to j_{\vec{a}}(r)$ as $k \to \infty$.

The next lemma describes how the function $j_{\vec{a}}$ interacts with the algebraic operations of addition and subtraction. Recall $\ominus$ from (2).

Lemma 2.4. *For $r, s \in \mathbb{R}_{\geq 0}$, $j_{\vec{a}}(r+s)$ and $j_{\vec{a}}(r \ominus s)$ are subsets of the union of the following three sets*

$$\begin{aligned} & j_{\vec{a}}(r) \cup j_{\vec{a}}(s), \\ & \{n \mid n+1 \in j_{\vec{a}}(r) \cup j_{\vec{a}}(s)\}, \\ & \{n \mid a_n \neq a_{n+1}\} \cup \{n \mid a_{n+1} \neq a_{n+2}\}. \end{aligned} \tag{5}$$

Proof. Suppose n is not in any of the sets in (5), that is, $r_n = r_{n+1} = r_{n+2}$, $s_n = s_{n+1} = s_{n+2}$ and $a_n = a_{n+1} = a_{n+2}$. By a direct computation of the digits of $r+s$ and $r \ominus s$ using the digits of r and s, we see that $(r+s)_n = (r+s)_{n+1}$ and $(r \ominus s)_n = (r \ominus s)_{n+1}$, so $n \notin j_{\vec{a}}(r+s)$ and $n \notin j_{\vec{a}}(r \ominus s)$. □

2.2.2. *Subrings of* $\mathbb{Q}$. Given a base sequence $\vec{a}$, we define a subring $\mathbb{Q}_{\vec{a}}$ of $\mathbb{Q}$. First, however, we make general comments on subrings of $\mathbb{Q}$. For an arbitrary set P of primes, define

$$P^{-1}\mathbb{Z} = \{\frac{k}{l} \mid k \in \mathbb{Z},\, l \in \mathbb{N}, \text{ and, for each prime } p, \text{ if } p \mid l, \text{ then } p \in P\}.$$

Note that $P^{-1}\mathbb{Z}$ is a subring of $\mathbb{Q}$. The following lemma is surely well known. We give an argument justifying it for the convenience of the reader.

Lemma 2.5. *Each subring of $\mathbb{Q}$ is of the form $P^{-1}\mathbb{Z}$, for a suitable set P of primes.*

Proof. Let R be a subring of $\mathbb{Q}$. It suffices to show that if $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ are relatively prime, $m/n \in R$, and a prime p divides n, then $1/p \in R$. To see this, note that n/p is in $\mathbb{N}$, and, by adding m/n to itself n/p many times, we get $m/p \in R$. Since $p \nmid m$, there are $a, b \in \mathbb{Z}$ such that $1 = ap + bm$. So we have

$$\frac{1}{p} = \frac{ap+bm}{p} = a + b\,\frac{m}{p},$$

from which we get $1/p \in R$, as required. □

Note that if we take $P = \emptyset$ in the lemma above, then $P^{-1}\mathbb{Z} = \mathbb{Z}$; if P is the set of all primes, then $P^{-1}\mathbb{Z} = \mathbb{Q}$. Furthermore, each subring of $\mathbb{Q}$ is Noetherian as each ideal in $P^{-1}\mathbb{Z}$ is of the form

$$n \cdot \big(P^{-1}\mathbb{Z}\big),$$

where $n = 0$ or n is a positive integer such that $p \nmid n$ for each $p \in P$; see [1, Proposition 3.11]. (We thank Patrick Allen for this reference.)

For a base sequence $\vec{a}$, let

$$\mathrm{pr}(\vec{a}) = \{p \mid \; p \text{ a prime and } p \mid a_n \text{ for all but finitely many } n\}.$$

Define

$$\mathbb{Q}_{\vec{a}} = \big(\mathrm{pr}(\vec{a})^{-1}\big)\mathbb{Z}.$$

We have the following immediate consequence of Lemma 2.5.

Proposition 2.6. *Each subring of $\mathbb{Q}$ is of the form $\mathbb{Q}_{\vec{a}}$ for some base sequence $\vec{a}$.*

2.3. **Ideals of subsets of $\mathbb{N}$.** Two properties of ideals of subsets of $\mathbb{N}$ will be important in the sequel: translation invariance and being an analytic P-ideal.

An ideal I is call **translation invariant** if, for each $x \in I$, we have

$$\{n+1 \mid n \in x\} \in I \;\text{ and }\; \{n \mid n+1 \in x\} \in I.$$

Before we talk about analytic P-ideals, we need to introduce appropriate submeasures. By a **lower semicontinuous** (**lsc**) **submeasure** we understand a lower semicontinuous function $\phi\colon 2^{\mathbb{N}} \to [0,\infty]$ such that $\phi(x) \leq \phi(y)$ if $x \subseteq y$ and $\phi(x \cup y) \leq \phi(x) + \phi(y)$ for all x, y. In this paper, we always assume that $0 < \phi(\{n\}) < \infty$ for each $n \in \mathbb{N}$. With each lsc submeasure we associate an ideal

$$\mathrm{Exh}(\phi) = \{x \subseteq \mathbb{N} \mid \phi(x \setminus \{1,\ldots,n\}) \to 0, \text{ as } n \to \infty\}.$$

An ideal I is an **analytic P-ideal** if it is analytic as a subsets of $2^{\mathbb{N}}$ and has the following property: for each sequence (x_n) of elements of I, there exists $y \in I$ such that $x_n \setminus y$ is finite for each n. By [7, Theorem 3.1], an ideal I is an analytic P-ideal if and only if it is of the form

$$I = \mathrm{Exh}(\phi),$$

for some lsc submeasure ϕ.

We have the following lemma on basic properties of ideals of the form $\mathrm{Exh}(\phi)$. All of these properties have been noticed in the literature. We recall them here for the reader's convenience.

Lemma 2.7. *Let ϕ be a lsc submeasure. Set $I = \mathrm{Exh}(\phi)$.*

(i) If ψ is a lsc submeasure with $I = \mathrm{Exh}(\psi)$, then, for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x \subseteq \mathbb{N}$ and $\phi(x) < \delta$, then $\psi(x) < \epsilon$.

(ii) For each $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \subseteq \mathbb{N}$ and $\phi(x), \phi(y) < \delta$, then $\phi(x \cup y) < \epsilon$.

(iii) If I is translation invariant, then, for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x \subseteq \mathbb{N}$ and $\phi(x) < \delta$, then

$$\phi\big(\{n \in \mathbb{N} \mid n+1 \in x\}\big) < \epsilon \;\text{ and }\; \phi\big(\{n+1 \in \mathbb{N} \mid n \in x\}\big) < \epsilon.$$

Proof. All three points are consequences of the following statements. For $\emptyset \neq x_n \subseteq \mathbb{N}$, $n \in \mathbb{N}$,

— if $\phi(x_n) \to 0$ as $n \to \infty$, then $\min_n x_n \to \infty$ as $n \to \infty$;
— if $\sum_n \phi(x_n) < \infty$, then $\bigcup_n x_n \in I$.

The first one of these statements follows from our assumption that $\phi(\{k\}) > 0$ for each $k \in \mathbb{N}$, while the second one is a consequence of semicontinuity of ϕ. □

2.4. **The module associated with a base sequence and an ideal.** Let $\vec{a}$ be a base sequence, which we fix for the duration of this section. Recall the definition (4) of $j_{\vec{a}}$, and set

$$j = j_{\vec{a}}.$$

We also fix an ideal I, to which the base sequence $\vec{a}$ is adapted. Define the following subset of $\mathbb{R}$

$$H = \{r \mid r \in \mathbb{R}_{\geq 0} \text{ and } j(r) \in I\} \cup \{-r \mid r \in \mathbb{R}_{\geq 0} \text{ and } j(r) \in I\}. \tag{6}$$

The set above will be the underlying subset of the Polish module we will define below.

We can immediately point out a complexity estimate on the set H that will be useful later on.

Lemma 2.8. *If I is an F_σ ideal, then H is an F_σ subset of $\mathbb{R}$.*

Proof. Set

$$\mathbb{R}_\infty = \{r \in \mathbb{R}_{\geq 0} \mid r \text{ has infinitely many non-zero digits}\}.$$

Obviously, it will suffice to show that the set

$$j^{-1}(I) = \{r \in \mathbb{R}_{\geq 0} \mid j(r) \in I\}$$

is F_σ. Since I is an F_σ ideal, we have that

$$I = \bigcup_n F_n$$

with $F_n \subseteq 2^{\mathbb{N}}$ closed. Let H_n be the closure in $\mathbb{R}_{\geq 0}$ of $j^{-1}(F_n)$. Directly from Lemma 2.3(i), we get

$$H_n \cap \mathbb{R}_\infty \subseteq j^{-1}(F_n).$$

It follows from this inclusion that

$$j^{-1}(I) \cap \mathbb{R}_\infty = \big(\bigcup_n j^{-1}(F_n)\big) \cap \mathbb{R}_\infty \subseteq (\bigcup_n H_n) \cap \mathbb{R}_\infty \subseteq \bigcup_n j^{-1}(F_n) = j^{-1}(I).$$

Thus, since $\mathbb{R}_{\geq 0} \setminus \mathbb{R}_\infty \subseteq j^{-1}(I)$ (as $j(r)$ is finite for each $r \in \mathbb{R}_{\geq 0} \setminus \mathbb{R}_\infty$), we have

$$j^{-1}(I) = \big(\bigcup_n H_n\big) \cup \big(\mathbb{R}_{\geq 0} \setminus \mathbb{R}_\infty\big).$$

Since $\mathbb{R}_{\geq 0} \setminus \mathbb{R}_\infty$ is countable and each H_n is closed, we see that $j^{-1}(I)$ is F_σ. □

2.4.1. *Algebraic operations on H.* In this section, we assume that the ideal I is translation invariant.

Lemma 2.9. *(i) H is a subgroup of $\mathbb{R}$ taken with addition $+$.*
(ii) H is closed under the multiplication by elements of $\mathbb{Q}_{\vec{a}}$.

Proof. Let

$$H_{\geq 0} = \{r \in \mathbb{R} \mid r \geq 0 \text{ and } j(r) \in I\}.$$

(i) Recall the operation $\ominus$ from (2). It suffices to show that $H_{\geq 0}$ is closed under $+$ and $\ominus$, which follows from Lemma 2.4 since $\vec{a}$ is adapted to I and the ideal I is translation invariant.

(ii) It is sufficient to show that for each $p \in \mathrm{pr}(\vec{a})$ and $r \in H_{\geq 0}$

$$\frac{1}{p} r \in H_{\geq 0}. \tag{7}$$

If $\frac{1}{p}r$ has finitely many non-zero digits, then $j(\frac{1}{p}r)$ is finite, so $\frac{1}{p}r \in H_{\geq 0}$. Thus, in the remainder of this proof, we assume that $\frac{1}{p}r$ has infinitely many non-zero digits.

Let $N \in \mathbb{N}$ such that $p \mid a_n$ for all $n \geq N$. The existence of such N follows from p belonging to $\mathrm{pr}(\vec{a})$. Now, (7) will be a consequence of the inclusion

$$j\big(\frac{1}{p}r\big) \subseteq j(r) \cup \{n+1 \mid n \in j(r)\} \cup \{n \mid a_n \neq a_{n+1}\} \cup \{1, \ldots, N\}, \tag{8}$$

since $\vec{a}$ is adapted to I, $j(r) \in I$, and I is translation invariant.

The proof of (8) is done by computing the digits of $\frac{1}{p}r$ from the digits of r. We need two pieces of notation. Let

$$[r]_n = [r] + \sum_{k=1}^{n} \frac{r_k}{a_1 \cdots a_k}, \tag{9}$$

and let r'_n be the integer such that

$$0 \leq r'_n < p \;\text{ and }\; r'_n \equiv r_n \mod p.$$

Also, the following identity, that is easy to see, will be useful

$$\frac{r_n}{a_1 a_2 \cdots a_n p} = \frac{\left[\frac{r_n}{p}\right]}{a_1 a_2 \cdots a_n} + \frac{\frac{a_{n+1}}{p} r'_n}{a_1 a_2 \cdots a_{n+1}}. \tag{10}$$

To get the relevant digits of $\frac{1}{p}r$, we first find certain digits of $\frac{1}{p}[r]_n$, for $n \geq N+1$. Observation (11) is verified by induction below:

$$\begin{gathered}\text{for } n \geq N+1, \text{ the } n\text{-th and } n+1\text{-st digits of } \tfrac{1}{p}[r]_n \text{ are} \\ \big[\frac{r_n}{p}\big] + \frac{a_n}{p} r'_{n-1} \text{ and } \frac{a_{n+1}}{p} r'_n, \text{ respectively,} \\ \text{and all the digits following them are } 0.\end{gathered} \tag{11}$$

Before we start proving (11), observe that a quick calculation shows

$$0 \leq \big[\frac{r_n}{p}\big] + \frac{a_n}{p} r'_{n-1} < a_n. \tag{12}$$

To obtain the base case $n = N+1$ of (11), note that $a_1 a_2 \cdots a_N \frac{1}{p}[r]_{N-1}$ is an integer, which implies that for all $n \geq N+1$, the n-th digit of $\frac{1}{p}[r]_{N-1}$ is 0. By adding $\frac{r_N}{a_1a_2\cdots a_N p}$ to $\frac{1}{p}[r]_{N-1}$ and using (10) with $n=N$, we see that $\frac{1}{p}[r]_N$ has the $N+1$-st digit $\frac{a_{N+1}}{p}r'_N$ and all the digits after that are 0. Using (10) with $n=N+1$ to add $\frac{r_{N+1}}{a_1a_2\cdots a_{N+1}p}$ to $\frac{1}{p}[r]_N$ and remembering (12), we obtain the base case. The inductive step is straightforward from the inductive hypothesis and (10).

From (11) and (10) we see that, for all $n \geq N+1$, adding $\frac{r_{n+1}}{a_1a_2\cdots a_{n+1}p}$ to $\frac{1}{p}[r]_n$ does not change the i-th digit for $i \leq n$. Putting together this observation and (11), we see that, for $n \geq N+1$, the n-th digit of $\frac{1}{p}[r]_k$ for $k \geq n$ is

$$\Big[\frac{r_n}{p}\Big] + \frac{a_n}{p}r'_{n-1}. \tag{13}$$

Since $\frac{1}{p}r$ is assumed to have infinitely many non-zero digits and $\frac{1}{p}[r]_k \to \frac{1}{p}r$ as $k \to \infty$, by Lemma 2.2(i), for each n, the n-th digit of $\frac{1}{p}r$ is equal to that of $\frac{1}{p}[r]_k$ for large k. In particular, for $n \geq N+1$, the n-th digit of $\frac{1}{p}r$ is given by (13). It follows that for $n \geq N+1$ if $r_{n-1}=r_n=r_{n+1}$ and $a_n=a_{n+1}$, then the n-th digit of $\frac{1}{p}r$ is the same as its $n+1$-st digit; that is, if $n-1 \notin j(r)$ and $n \notin j(r)$ and $a_n = a_{n+1}$, then $n \notin j\big(\frac{1}{p}r\big)$, and (8) follows. $\square$

2.4.2. *Topology on H.* In this section, we assume that the ideal I is translation invariant and an analytic P-ideal.

Our goal is to describe a topology on H. We will need the following general lemma that is essentially due to Chittenden [2]. We derive it here from the basic theory of uniformities, for which we refer the reader to [3, Section 8.1].

Lemma 2.10. *Let X be a set and $\rho\colon X\times X \to \mathbb{R}$ a function such that*

(i) for all $x,y \in X$, $\rho(x,y)\geq 0$, and $\rho(x,y)=0$ if and only if $x=y$;
(ii) for all $x,y\in X$, $\rho(x,y)=\rho(y,x)$;
(iii) for each $\epsilon>0$, there exists $\delta>0$ such that, for all $x,y,z\in X$, if $\rho(x,y)<\delta$ and $\rho(y,z)<\delta$, then $\rho(x,z)<\epsilon$.

Then there exists a metric d on X such that, for each $\epsilon>0$, there exists $\delta>0$ with

$$\big(\rho(x,y)<\delta \Rightarrow d(x,y)<\epsilon\big) \text{ and } \big(d(x,y)<\delta \Rightarrow \rho(x,y)<\epsilon\big), \text{ for all } x,y\in X.$$

Proof. For $A\subseteq X\times X$, let

$$A^{-1} = \{(y,x) \mid (x,y)\in A\}.$$

Define

$$V_\epsilon = \{(x,y)\in X\times X \mid \rho(x,y)<\epsilon\}, \text{ for } \epsilon>0. \tag{14}$$

The family $\mathcal{U}$ consisting of all subsets A of $X\times X$ such that $A=A^{-1}$ and $A\supseteq V_\epsilon$ for some $\epsilon>0$ is a uniformity on X. To show this, we need to see the following three properties:

(i) $\bigcap_{\epsilon>0} V_\epsilon = \{(x,y)\in X\times X \mid x=y\}$;
(ii) $V_\epsilon^{-1} = V_\epsilon$;
(iii) for each $\epsilon>0$, there exists $\delta>0$ such that

$$\{(x,z) \mid (x,y),(y,z)\in V_\delta, \text{ for some } y\in X\}\subseteq V_\epsilon.$$

These properties are immediate corollaries of the corresponding properties of ρ from the assumptions of the lemma. By [3, Theorem 8.1.21], the uniformity $\mathcal{U}$ is given by a metric d on X. The conclusion of the lemma follows from the definition of the sets V_ϵ. $\square$

Since I is an analytic P-ideal, we have $I = \mathrm{Exh}(\phi)$, for some lsc submeasure ϕ. We fix such ϕ. Recall the operation $\ominus$ from (2), and define

$$\rho(r, s) = \phi\big(j(r \ominus s)\big) + (r \ominus s), \quad \text{for } r, s \in \mathbb{R}.$$

The next result gives a weak form of the triangle inequality for ρ as in Lemma 2.10(iii).

Lemma 2.11. *For any $\epsilon > 0$ there exists $\delta > 0$ such that for any r, s, t in H,*

$$\rho(r, s),\ \rho(s, t) < \delta \Rightarrow \rho(r, t) < \epsilon.$$

Proof. Fix $\epsilon > 0$. Note that $r \ominus t$ is equal to one of the following

$$(r \ominus s) + (s \ominus t),\ (r \ominus s) \ominus (s \ominus t).$$

By Lemma 2.4, $j(r \ominus t)$ is included in the union of the following three sets

$$\begin{aligned} & j(r \ominus s) \cup j(s \ominus t), \\ & \{n \mid n+1 \in j(r \ominus s) \cup j(s \ominus t)\}, \\ & \{n \mid a_n \neq a_{n+1}\} \cup \{n \mid a_{n+1} \neq a_{n+2}\}. \end{aligned} \tag{15}$$

By Lemma 2.7(ii) and (iii), there is $\delta > 0$ such that

$$\phi\big(j(r \ominus s)\big),\ \phi\big(j(s \ominus t)\big) < \delta \Rightarrow \phi\big(\{n \mid n+1 \in j(r \ominus s) \cup j(s \ominus t)\}\big) < \epsilon. \tag{16}$$

Moreover, since $\{n \mid a_n \neq a_{n+1}\} \cup \{n \mid a_{n+1} \neq a_{n+2}\}$ is in I, there is N large enough such that

$$\phi\big(\{n \mid a_n \neq a_{n+1}\} \cup \{n \mid a_{n+1} \neq a_{n+2}\} \setminus \{1, \ldots, N\}\big) < \epsilon. \tag{17}$$

For this N, by Lemma 2.1(i), there exits $\gamma > 0$ such that if $0 \leq r < \gamma$, then the first $N+2$ digits of r are all zeros. It follows that if $r \ominus s,\ s \ominus t < \gamma$, then the first $N+2$ digits of $r \ominus s$ and $s \ominus t$ are all zeros, therefore, $r \ominus t$ has the first $N+1$ digits equal to zero. This implies that $j(r \ominus t)$ contains no elements from $\{1, \ldots, N\}$. Furthermore, from the definition of $\ominus$, we have

$$r \ominus s < \epsilon,\ s \ominus t < \epsilon \Rightarrow r \ominus t < 2\epsilon. \tag{18}$$

Finally, let

$$\delta' = \min\{\gamma, \delta, \epsilon\}.$$

Hence, by our choice of δ', (16), (17), and (18), we get

$$\rho(r, s),\ \rho(s, t) < \delta' \Rightarrow \rho(r, t) < 6\epsilon,$$

and the lemma follows. $\square$

Lemma 2.12. *There exists a metric d on H such that for each $\epsilon > 0$, there exists $\delta > 0$ with*

$$\big(\rho(r, s) < \delta \Rightarrow d(r, s) < \epsilon\big) \textit{ and } \big(d(r, s) < \delta \Rightarrow \rho(r, s) < \epsilon\big),\ \textit{for all } r, s \in H.$$

Proof. It suffices to check that ρ fulfills properties (i)–(iii) from the assumption of Lemma 2.10. It is clear that ρ has (i) and (ii); ρ satisfies (iii) by Lemma 2.11. $\square$

Consider the topology induced on H by the metric d from Lemma 2.12. We call this topology the **submeasure topology**. The definitions of ρ, d, and, therefore, also the submeasure topology depend on ϕ with $I = \mathrm{Exh}(\phi)$. Lemma 2.7(i) implies that these objects depend only on I and not on the choice of ϕ.

2.4.3. *The Polish module.* Recall that we have a base sequence $\vec{a}$ and an ideal I such that $\vec{a}$ is adapted to I. We assume now that I is a translation invariant analytic P-ideal I. We are ready to define the $\mathbb{Q}_{\vec{a}}$-module

$$I[\vec{a}].$$

The underlying set of $I[\vec{a}]$ is H from (6). By Lemma 2.9, H with the operation of $+$ and multiplication by elements of $\mathbb{Q}_{\vec{a}}$ inherited from $\mathbb{R}$ is a $\mathbb{Q}_{\vec{a}}$-module. We topologize H with the submeasure topology.

Theorem 2.13. *Let I be a translation invariant analytic P-ideal of subsets of $\mathbb{N}$, and let a base sequence $\vec{a}$ be adapted to I. Then $I[\vec{a}]$ is a Polish $\mathbb{Q}_{\vec{a}}$-module and the identity map $I[\vec{a}] \to \mathbb{R}$ is a continuous $\mathbb{Q}_{\vec{a}}$-embedding.*

We isolate the following result as a lemma as we will use it twice.

Lemma 2.14. *Let F be the collection of elements of $\mathbb{R}_{\geq 0}$ that have finitely many non-zero digits. If I has an element that is an infinite subset of $\mathbb{N}$, then $F \cup -F$ is dense in $I[\vec{a}]$ with the submeasure topology.*

Proof. It suffices to show that F is dense in $H_{\geq 0}$. For $r \in H_{\geq 0}$, recall $[r]_n$ from (9). Clearly $[r]_n$ is in F for every n. We verify that there is a subsequence of $([r]_n)_{n\in\mathbb{N}}$ that converges to r in the submeasure topology. If r is in F, we have $[r]_n = r$ for all but finitely many n. If r is not in F, there are two cases.

When $j(r)$ is finite, fix a sequence (n_i) of natural numbers such that $\phi(\{n_i\}) \to 0$. The existence of such a sequence follows from the assumption that an infinite subset of $\mathbb{N}$ is in I. Since $j(r - [r]_{n_i}) = \{n_i\}$ for all but finitely many i,

$$\phi\big(j(r - [r]_{n_i})\big) \to 0 \ \text{ as } i \to \infty. \tag{19}$$

Now assume $j(r)$ is infinite. Note that, for $n \in j(r)$, $j(r - [r]_n)$ is equal to one of the following two sets

$$j(r) \setminus \{1, \ldots, n-1\},\ j(r) \setminus \{1, \ldots, n\}$$

Thus, since $j(r) \in \mathrm{Exh}(\phi)$, we get

$$\phi\big(j(r - [r]_n)\big) \to 0 \ \text{ as } j(r) \ni n \to \infty. \tag{20}$$

Furthermore, $(r - [r]_n) \to 0$ in $\mathbb{R}$ as $n \to \infty$, so, in both cases, by (19) and (20), we get a subsequence of $([r]_n)$ that converges to r in terms of ρ, and this implies convergence in the submeasure topology by Lemma 2.12. $\square$

Proof of Theorem 2.13. We need to see that the submeasure topology on $I[\vec{a}]$ is Polish. If I is the ideal of all finite subsets of $\mathbb{N}$, then $I[\vec{a}]$ is countable and the submeasure topology is discrete, so $I[\vec{a}]$ is Polish. The first observation is immediate. To see the second one, fix a lsc submeasure ϕ such that $I = \mathrm{Exh}(\phi)$ and $\phi(\{n\}) > 1$ for all n. As pointed out, the submeasure topology does not depend on the choice of ϕ. Let $\epsilon = \frac{1}{a_1}$. Then, for $r, s \in H$ with $\rho(r,s) < \epsilon$, we can conclude from the definition of ρ and Lemma 2.1(i) that $[r \ominus s] = 0$, the first digit of $r \ominus s$ is 0, and $j(r \ominus s) = \emptyset$. These properties together imply that $r \ominus s = 0$. By Lemma 2.12, this implies the topology induced by d is discrete.

Now we assume I is not the ideal of finite sets, or equivalently, I has an infinite subset of $\mathbb{N}$ as element. Separability follows from Lemma 2.14. It remains to see that the metric d is complete, that is, we need to see that each d-Cauchy sequence in $I[\vec{a}]$ converges with respect to d to an element of $I[\vec{a}]$. It will suffice to do it for sequences contained in $H_{\geq 0}$ and, by Lemma 2.12, with Cauchy-ness and convergence understood in terms of ρ. So let $(r(n))$ be a sequence in $H_{\geq 0}$ such that, for each $\epsilon > 0$, $\rho(r(m), r(n)) < \epsilon$ for large m, n. By the definition of ρ, $(r(n))$ is Cauchy in $\mathbb{R}$, so it converges to some $r \in \mathbb{R}_{\geq 0}$. We need to show that $r \in H_{\geq 0}$ and $\rho(r(n), r) \to 0$. To do that, we first prove the following claim.

Claim. Given $\epsilon > 0$, for each large enough $M \in \mathbb{N}$, we have

$$\phi\big(j(r(n)) \setminus \{1, \ldots, M\}\big) < \epsilon \ \text{ for large } n.$$

Proof of Claim. Using Lemma 2.11, for the given ϵ, we fix $\delta > 0$ such that for all $s, t, u \in H_{\geq 0}$

$$\rho(s,t),\ \rho(t,u) < \delta \Rightarrow \rho(s,u) < \epsilon/2. \tag{21}$$

By Cauchy-ness of $(r(n))$, there exists k such that

$$\rho(r(n), r(k)) < \delta \text{ for all } n \geq k.$$

Since $r(k)$ is in $H_{\geq 0}$ and, from Lemma 2.14, F is dense in $H_{\geq 0}$, there is a number s with finitely many non-zero digits such that

$$\rho(r(k), s)) < \delta.$$

By (21) and the two inequalities above, $\rho(r(n), s) < \epsilon/2$ for all $n \geq k$. In particular,

$$\phi\big(j(r(n) \ominus s)\big) < \frac{\epsilon}{2}, \text{ for all } n \geq k. \tag{22}$$

Notice that, for each n, either $r(n) \ominus s = r(n) - s$ or $r(n) \ominus s = s - r(n)$. In the first case, it is easy to see that subtracting s only affects finitely many digits of $r(n)$, so, for large enough M, which depends only on s,

$$j\big(r(n)\big) \setminus \{1, \ldots, M\} \subseteq j\big(r(n) - s\big). \tag{23}$$

In the second case, it is not hard to see that for large enough M, which also only depends on s,

$$j\big(r(n)\big) \setminus \{1, \ldots, M\} \subseteq j\big(s - r(n)\big) \cup \{n \mid a_n \neq a_{n+1}\}. \tag{24}$$

Further, since $\{n \mid a_n \neq a_{n+1}\} \in I$, there is $N \in \mathbb{N}$ such that

$$\phi\big(\{n \mid a_n \neq a_{n+1}\} \setminus \{1,\dots,N\}\big) < \epsilon/2.$$

Hence, putting together both cases, we see that for M such that (23) and (24) hold and, in the second case, $M \geq N$, we have

$$\phi\big(j(r(n)) \setminus \{1,\dots,M\}\big) < \phi\big(j(r(n) \ominus s)\big) + \epsilon/2.$$

This inequality implies the conclusion of the claim by (22) .

For contradiction, suppose that $r \notin H_{\geq 0}$, that is, $j(r) \notin I$ since r is non-negative. Then there exists $\epsilon > 0$ such that, for each m, there is a finite set $P_m \subseteq j(r)$ with $\min(P_m) > m$ and $\phi(P_m) \geq \epsilon$. By Lemma 2.3(i) and the fact that r has infinitely many non-zero digits (as otherwise $r \in H_{\geq 0}$), each P_m is included in $j(r(n))$ for large enough n, which contradicts Claim when $m = M$. Hence r is in $H_{\geq 0}$.

We now show that $\rho(r(n), r) \to 0$. Since $r(n) \ominus r \to 0$ in $\mathbb{R}$ as $n \to \infty$, it is enough to see that

$$\phi\big(j(r(n) \ominus r)\big) \to 0 \text{ as } n \to \infty. \tag{25}$$

Assume towards a contradiction that this is not the case, so we can fix $\epsilon > 0$ with

$$\phi\big(j(r(n) \ominus r)\big) > \epsilon \text{ for infinitely many } n. \tag{26}$$

By Lemma 2.1(i), we have that for each i, for large n, $(r(n) \ominus r)_i = 0$. Thus, we get that, for each M,

$$\{1,\dots,M\} \cap j\big(r(n) \ominus r\big) = \emptyset \;\text{ for large } n. \tag{27}$$

By Lemma 2.4, for each n, $j(r(n) \ominus r)$ is included in the union of the following sets

$$\begin{aligned}&j(r(n)),\; j(r),\; \{k \mid a_k \neq a_{k+1}\},\\ &\{k \mid k+1 \in j\big(r(n)\big)\},\; \{k \mid k+1 \in j(r)\},\; \{k \mid a_{k+1} \neq a_{k+2}\}.\end{aligned}$$

Thus, by (26), (27), and Lemma 2.7(iii), there exists $\delta > 0$ such that, for each M, one of the following statements is true

$$\begin{gathered}\phi\big(j(r(n)) \setminus \{1,\dots,M\}\big) > \delta \;\text{ for infinitely many } n,\\ \phi\big(j(r) \setminus \{1,\dots,M\}\big) > \delta,\\ \phi\big(\{k \mid a_k \neq a_{k+1}\} \setminus \{1,\dots,M\}\big) > \delta.\end{gathered} \tag{28}$$

Since $j(r), \{k \mid a_k \neq a_{k+1}\} \in I$, for large M, the last two inequalities of (28) fail. Hence, the first inequality holds for large M contradicting Claim. So (25) is proved.

Note that, by Lemma 2.12 and the definition of ρ, the submeasure topology on $I[\vec{a}]$ contains the topology it inherits from $\mathbb{R}$. Thus, the inclusion map $I[\vec{a}] \to \mathbb{R}$ is continuous. Since the algebraic operations on H are inherited from $\mathbb{R}$, the inclusion map is a $\mathbb{Q}_{\vec{a}}$-embedding.

We prove that $I[\vec{a}]$ with $+$ is a topological group with the submeasure topology. Since the submeasure topology is Polish, by [5, Corollary 9.15], it suffices to show that the functions

$$I[\vec{a}] \ni s \to r + s \in I[\vec{a}],\; \text{ for each } r \in I[\vec{a}]$$

and

$$I[\vec{a}] \ni s \to -s \in I[\vec{a}]$$

are continuous. Observe that ρ has the following invariance properties

$$\rho(s_1, s_2) = \rho(r + s_1, r + s_2) \text{ and } \rho(s_1, s_2) = \rho(-s_1, -s_2)$$

for all $r, s_1, s_2 \in I[\vec{a}]$. The continuity of the functions above is now an immediate consequence of the invariance properties of ρ and Lemma 2.12.

Finally, we show that multiplication by elements of $\mathbb{Q}_{\vec{a}}$ is continuous on $I[\vec{a}]$ taken with the submeasure topology. Observe first that since the submeasure topology is Polish and the identity map $I[\vec{a}] \to \mathbb{R}$ is continuous, by [5, Corollary 15.2], the submeasure topology has the same Borel sets as the topology $I[\vec{a}]$ inherited from $\mathbb{R}$. Now, multiplication by an element of $\mathbb{Q}_{\vec{a}}$, being a Borel map on $I[\vec{a}]$ taken with the topology inherited from $\mathbb{R}$, is also a Borel map on $I[\vec{a}]$ taken with the submeasure topology. It is clearly also a group homomorphism. Thus, since the submeasure topology is Polish, by Pettis Theorem [5, Theorem 9.10], this map is continuous. □

3. Inclusions and homomorphisms among modules

We will need two more properties of ideals and base sequences. An ideal of subsets of $\mathbb{N}$ is **tall** if each infinite subset of $\mathbb{N}$ contains an infinite subset in I. If I is of the form $I = \mathrm{Exh}(\phi)$ for a lsc submeasure ϕ, then it is easy to check that I is tall precisely when

$$\phi\big(\{n\}\big) \to 0 \text{ as } n \to \infty. \tag{29}$$

We call a base sequence $\vec{a}$ **uniform** provided that, for each prime p, if $p \mid a_n$ for some n, then $p \mid a_n$ for all but finitely many n; in other words,

$$\mathrm{pr}(\vec{a}) = \{p \mid \ p \text{ a prime and } p \mid a_n \text{ for some } n\}.$$

3.1. Inclusions and non-inclusions.

In this section, we view modules of the form $I[\vec{a}]$ simply as subsets of $\mathbb{R}$. We are interested in the inclusion relation among such sets as the ideal I varies.

First, we have an observation that inclusion between ideals implies inclusion between the corresponding modules.

Proposition 3.1. *Let I, J be translation invariant analytic P-ideals, and let $\vec{a}$ be a base sequence adapted to both I and J. If $I \subseteq J$, then $I[\vec{a}] \subseteq J[\vec{a}]$.*

Proof. The inclusion $I[\vec{a}] \subseteq J[\vec{a}]$ follows directly from the assumption $I \subseteq J$. □

Now, we prove a theorem asserting that a slightly stronger notion of non-inclusion between ideals implies non-inclusion between the corresponding modules, even if we allow the modules to be multiplied by non-zero real numbers. We need the following two definitions. Let I and J be analytic P-ideals. We say that I **is not included in J on intervals** if there are lsc submeasures ϕ and ψ with $I = \mathrm{Exh}(\phi)$ and $J = \mathrm{Exh}(\psi)$, for which there exists $d > 0$ such that

$$\inf\{\phi(P) \mid P \text{ an interval in } \mathbb{N} \text{ with } \psi(P) \geq d\} = 0.$$

Notice that this property implies $I \not\subseteq J$.

For a subset X of $\mathbb{R}$ and a real number c, we write

$$cX = \{cx \mid x \in X\}.$$

Theorem 3.2. *Let I, J be translation invariant, analytic P-ideals with I being tall. Let $\vec{a}$ be a base sequence adapted to both I and J. If I is not included in J on intervals, then $c\,I[\vec{a}] \not\subseteq J[\vec{a}]$, for each non-zero real number c.*

Proof. Since $I[\vec{a}]$ is closed under addition and taking additive inverses, it will suffice to show the conclusion only for $c \geq 1$. So we assume $c \geq 1$, and set

$$C = c\,I[\vec{a}],$$

with the aim to show that $C \not\subseteq J[\vec{a}]$.

Since I is not included in J on intervals, there are lsc submeasures ϕ and ψ such that $I = \mathrm{Exh}(\phi)$ and $J = \mathrm{Exh}(\psi)$, for which there exist $d > 0$ and a sequence of intervals $P_n = [l_n, m_n]$, with $m_n < l_{n+1}$, such that

$$\psi(P_n) > d \;\text{ and }\; \phi(P_n) < 2^{-n}. \tag{30}$$

We also set $j = j_{\vec{a}}$.

Claim 1. Given $n \in \mathbb{N}$, there is $y \in C$ such that

(i) $j(y) \supseteq P_n$;
(ii) $0 \leq y < \frac{1}{a_1 a_2 \cdots a_{l_n}}$;
(iii) $y_{m_n+2} \neq 0$.

Proof of Claim 1. To find such y, pick $z \in [0, 1)$ so that

$$j(z) \cap [1, m_n] = P_n;\; z_{m_n+2} \neq 0;\; z_i = 0, \text{ for all } i \in \mathbb{N} \setminus [l_n + 1, m_n + 2]. \tag{31}$$

Note that from (31), we have

$$0 < z < \frac{1}{a_1 a_2 \cdots a_{l_n}}. \tag{32}$$

Since C is dense in $\mathbb{R}$, we can approximate z from above by elements y of C. For a tight enough such approximation y, we get (ii) from (32), (i) from (31) and Lemma 2.3(ii), and (iii) from (31) and Lemma 2.2(ii). The claim follows.

Claim 2. Given $\epsilon > 0$, for large enough $n \in \mathbb{N}$ the following statement holds. For each $v \in I[\vec{a}]$ with $0 \leq v < \frac{1}{a_1 a_2 \cdots a_{l_n}}$, there exists $w \in I[\vec{a}]$ such that

(i) $0 \leq w \leq v < w + \frac{1}{c\, a_1 a_2 \cdots a_{m_n+2}}$;
(ii) $\phi\big(j(w)\big) < \epsilon$.

Proof of Claim 2. Let $k_0 \in \mathbb{N}$ be such that $2^{k_0} > c$. For large enough n, we have

$$2^{-n} < \frac{\epsilon}{3} \tag{33}$$

and, by tallness of I applied using condition (29),

$$\phi(X) < \frac{\epsilon}{3}, \text{ for each set } X \text{ of size } \leq k_0 + 3 \text{ with } i > m_n \text{ for all } i \in X. \tag{34}$$

Fix n with (33) and (34). Since $v \in I[\vec{a}]$ and $v \geq 0$, we have $j(v) \in I$, so there is $N \in \mathbb{N}$ with $N > m_n + k_0 + 2$ and such that

$$\phi\big(j(v) \setminus [1, N]\big) < \epsilon/3. \tag{35}$$

Let

$$K = [m_n + 1,\ m_n + k_0 + 2],$$

and define w to be the real number such that

$$w_i = \begin{cases} v_i, & \text{if } i \leq m_n + k_0 + 2 \text{ or } i \geq N + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since v and w differ on finitely many digits and $v \in I[\vec{a}]$, it follows from the definition of the underlying set of $I[\vec{a}]$ that $w \in I[\vec{a}]$.

We check that w satisfies the two properties in the claim. First, directly from the definition we have

$$0 \leq v - w < \frac{1}{a_1 \cdots a_{m_n+2} \cdots a_{m_n+k_0+2}} \leq \frac{1}{a_1 \cdots a_{m_n+2}\, 2^{k_0}}.$$

from which, by our choice of k_0, we get (i). Second, since $0 \leq w \leq v < \frac{1}{a_1 a_2 \cdots a_{l_n}}$, using Lemma 2.1(i), we see that the first l_n digits of w are 0. It follows that

$$j(w) \cap [1, l_n - 1] = \emptyset. \tag{36}$$

Note that $j(w) \cap [m_n + 1, N]$ is a subset of $K \cup \{N\}$. Hence, by (34), we get

$$\phi\big(j(w) \cap [m_n + 1, N]\big) \leq \phi\big(K \cup \{N\}\big) < \epsilon/3. \tag{37}$$

From the definition of w and by (35), we have

$$\phi\big(j(w) \setminus [1, N]\big) = \phi\big(j(v) \setminus [1, N]\big) < \epsilon/3. \tag{38}$$

Putting together (36), (37) and (38), we get

$$\phi(j(w)) < 0 + \phi(P_n) + \epsilon/3 + \epsilon/3 = \phi(P_n) + 2\epsilon/3.$$

The above inequality implies the conclusion of the claim by (30) and (33).

We use Claims 1 and 2 to show that given $\epsilon > 0$, there exists $w \in I[\vec{a}]$ such that

(a) $0 \leq w < \epsilon$;
(b) $\phi\big(j(w)\big) < \epsilon$;
(c) $\psi\big(j(c\,w)\big) > d$.

Let $\epsilon > 0$. Fix n large enough so the conclusion of Claim 2 holds and

$$\frac{1}{a_1 a_2 \cdots a_{l_n}} < \epsilon. \tag{39}$$

For this n, Claim 1 produces $y \in C$ with properties (i)–(iii) from that claim. Set

$$v = \frac{y}{c}.$$

Note that $v \in I[\vec{a}]$ since $y \in C$. Using the inequality $c \geq 1$ and Claim 1(ii), we get

$$0 \leq v \leq y < \frac{1}{a_1 a_2 \cdots a_{l_n}}. \tag{40}$$

So Claim 2 applies to v producing $w \in I[\vec{a}]$ with properties (i) and (ii) from that claim.

Since, by Claim 2(i), $0 \leq w \leq v$, we get (a) from (39) and (40). Point (b) is immediate from Claim 2(ii). By Claim 2(i), we have

$$0 \leq c\,w \leq y < c\,w + \frac{1}{a_1 a_2 \cdots a_{m_n+2}}.$$

These inequalities imply (c) by Claim 1(i) and (iii), Lemma 2.1(ii), and (30).

Now, to finish the proof of the theorem, assume for contradiction that $C \subseteq J[\vec{a}]$. Then $x \to cx$ defines a function from $I[\vec{a}]$ to $J[\vec{a}]$. The map $x \to cx$, being a Borel map from $\mathbb{R}$ to $\mathbb{R}$, is also a Borel map from $I[\vec{a}]$ to $J[\vec{a}]$ both taken with the submeasure topologies by [5, Corollary 15.2]. Moreover, it is a group homomorphism, so by Pettis Theorem [5, Theorem 9.10], it is continuous. However, the existence of $w \in I[\vec{a}]$ for each $\epsilon > 0$ with properties (a)–(c) implies, by Lemma 2.12, that this map is discontinuous at 0, yielding a contradiction. $\square$

3.2. **Consequences concerning module homomorphisms.** We derive some consequences of Theorem 3.2 for homomorphisms among modules constructed in Section 2. We will use these consequences to prove Theorem 1.1.

Lemma 3.3. *Let I, J be translation invariant, analytic P-ideals, with I being tall. Let $\vec{a}$ be a uniform base sequence adapted to both I and J. If I is not included in J on intervals, then each continuous $\mathbb{Q}_{\vec{a}}$-module homomorphism from $I[\vec{a}]$ to $J[\vec{a}]$ is identically equal to* 0.

To prove Lemma 3.3, we need a result on the form of continuous module homomorphisms.

Lemma 3.4. *Let I, J be translation invariant, analytic P-ideals, with I having an infinite subset of $\mathbb{N}$ as element. Let $\vec{a}$ be a uniform base sequence adapted to I and J. If $f\colon I[\vec{a}] \to J[\vec{a}]$ is a continuous $\mathbb{Q}_{\vec{a}}$-module homomorphism, then there exists $c \in \mathbb{R}$ with $f(y) = c\,y$ for all $y \in I[\vec{a}]$.*

Proof. We begin with showing that $\mathbb{Q}_{\vec{a}}$ is dense in $I[\vec{a}]$. Let F be the collection of elements of $\mathbb{R}_{\geq 0}$ that have finitely many non-zero digits. By Lemma 2.14, $F \cup -F$ is dense in $I[\vec{a}]$. We show $F \cup -F = \mathbb{Q}_{\vec{a}}$.

Suppose r is non-negative and in $\mathbb{Q}_{\vec{a}}$, that is, $r = \frac{k}{l}$, where $k \in \mathbb{N}$ and l is a product of primes in $\mathrm{pr}(\vec{a})$. Each prime in $\mathrm{pr}(\vec{a})$ divides a_n for all but finitely many n. Thus, there is N such that $l \mid a_1 a_2 \cdots a_N$. Therefore, $a_1 a_2 \cdots a_N r$ is an integer, which implies that $r_n = 0$ for all $n > N$. It follows that r is in F. By the same argument, we see that if $r \in \mathbb{Q}_{\vec{a}}$ is non-positive, then r is in $-F$.

Conversely, if r is in $F \cup -F$, there exists N such that $a_1 a_2 \cdots a_N r$ is an integer. Hence r is of the form $\frac{k}{a_1 a_2 \cdots a_N}$. Each prime p that divides $a_1 a_2 \cdots a_N$ divides some a_n, so p is in $\mathrm{pr}(\vec{a})$ by the uniformity of $\vec{a}$. Therefore, r is in $\mathbb{Q}_{\vec{a}}$.

We continue our proof of Lemma 3.4. Note that 1 is in $I[\vec{a}]$ since 1 has finitely many non-zero digits. Set

$$c = f(1).$$

Note that for each $y \in \mathbb{Q}_{\vec{a}}$, we have

$$f(y) = f(y1) = yf(1) = c\,y, \tag{41}$$

because f is a $\mathbb{Q}_{\vec{a}}$-module homomorphism.

Since f is continuous as a function from $I[\vec{a}]$ to $J[\vec{a}]$, and the topology on $J[\vec{a}]$ contains the topology it inherits from $\mathbb{R}$, f is continuous as a function from $I[\vec{a}]$ to $\mathbb{R}$. Similarly, since the topology on $I[\vec{a}]$ contains the topology it inherits from $\mathbb{R}$, the function $y \to c\,y$ is continuous as a function from $I[\vec{a}]$ to $\mathbb{R}$. It follows that (41) holds for all y in the closure of $\mathbb{Q}_{\vec{a}}$ in $I[\vec{a}]$, which gives the lemma as $\mathbb{Q}_{\vec{a}}$ is dense in $I[\vec{a}]$. $\square$

Proof of Lemma 3.3. The conclusion of the corollary follows from Theorem 3.2 and Lemma 3.4. $\square$

4. Proof of Theorem 1.1

We construct a family I_x, $x \subseteq \mathbb{N}$, of analytic P-ideals that will allow us to apply Lemma 3.3 to obtain Theorems 1.1. This family is probably the simplest one that can serve our purpose, but some more complicated families from the literature would also work; see, for example, [6].

For $k \in \mathbb{N}$, let

$$P_k = \{n \in \mathbb{N} \mid 2^{k-1} \leq n < 2^k\}.$$

We note the following properties of the intervals P_k:

(a) $\sum_{n\in P_k} \frac{1}{n} \geq \frac{1}{2}$;

(b) $\sum_{n\in a} \frac{1}{n} < \infty$, for each $a \subseteq \mathbb{N}$ that has at most one point in common with P_k for each k.

For $x \subseteq \mathbb{N}$, set $A_x = \bigcup_{k\in x} P_k$, and let, for $a \subseteq \mathbb{N}$,

$$\phi_x(a) = \sum_{n\in a\cap A_x} \frac{1}{2^n} + \sum_{n\in a\setminus A_x} \frac{1}{n}.$$

It is clear that ϕ_x is a lsc submeasure. Finally, define

$$I_x = \mathrm{Exh}(\phi_x)$$

and observe that

$$I_x = \{a \subseteq \mathbb{N} \mid \phi_x(a) < \infty\}.$$

It is then clear that I_x is an analytic P-ideal and, using (b), that it is translation invariant and tall. Furthermore, observe that, for $x, y \subseteq \mathbb{N}$,

$$x \setminus y \text{ finite} \Longrightarrow I_x \subseteq I_y, \tag{42}$$

$$x \setminus y \text{ infinite} \Longrightarrow I_x \text{ is not included in } I_y \text{ on intervals.} \tag{43}$$

The first implication is clear since the assumption gives that $A_x \setminus A_y$ is finite. To see the second implication, note that if $x \setminus y$ is infinite, then the sequence of intervals $(P_k)_{k\in x\setminus y}$ witnesses the non-inclusion on intervals as, by (a), $\phi_y(P_k) \geq 1/2$ for $k \in x \setminus y$, while $\phi_x(P_k) \to 0$ for $x \setminus y \ni k \to \infty$.

Proof of Theorem 1.1. Let R be a subring of $\mathbb{Q}$ not equal to $\mathbb{Z}$; that is,

$$R = P^{-1}\mathbb{Z}$$

for a nonempty set P of primes. It is easy to find a uniform base sequence $\vec{\mathbf{a}}$ with

$$P = \mathrm{pr}(\vec{a}) \text{ and } \{k \mid a_k \neq a_{k+1}\} \subseteq \{2^k \mid k \in \mathbb{N}\}.$$

Note that the first equality gives $R = \mathbb{Q}_{\vec{a}}$ and the second one ensures that $\vec{a}$ is adapted to I_x for each $x \subseteq \mathbb{N}$.

Then, for each $x \subseteq \mathbb{N}$, $I_x[\vec{a}]$ is a Polish R-module by Theorem 2.13. Let $f_x \colon I_x[\vec{a}] \to \mathbb{R}$ be the identity map. This is a continuous R-embeding into $\mathbb{R}$. Note that since each I_x is F_σ, it follows from Lemma 2.8 that $I_x[\vec{a}]$ is an F_σ as a subset of $\mathbb{R}$.

Observe that, by Proposition 3.1 and (42), if $x \setminus y$ is finite, $x, y \subseteq \mathbb{N}$, then $I_x[\vec{a}] \subseteq I_y[\vec{a}]$ as subsets of $\mathbb{R}$. The identity map $I_x[\vec{a}] \to I_y[\vec{a}]$ is obviously an R-module homomorphism and it is continuous by Pettis Theorem [5, Theorem 9.10]. If $x \setminus y$ is infinite, then the conclusion follows from (43) and Lemma 3.3. □

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