A NOTE ON PRO-LIE GROUPS

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Abstract. We give a short proof of the theorem that a closed subgroup of a countable product of second countable Lie groups is pro-Lie.

The point of this note is to give a short and self-contained, modulo well known results, proof of a theorem of Hofmann and Morris [3] (see also [4, Theorem 3.35] and [5]) in the case of second countable groups. Another simple proof of the result of Hofmann and Morris was found by Glöckner [2]. All the ideas in the argument which we present here come from the proof of Lemma 2.3 in our paper [6]. We thank Isaac Goldbring for pointing out to us possible connection between our considerations in [6] and pro-Lie groups.

All groups below are assumed to be second countable.

1. Common knowledge background

In Proposition 1.1 we collect some well known properties of Lie groups that will be used. Important to us will be the notion of dimension of a Lie group, which can be understood as the dimension of the underlying manifold.

Proposition 1.1. (i) Connected components of a Lie group are open and the connected component of the identity is a Lie group.

(ii) If M is a Lie group and N a closed subgroup of M, then N is a Lie group; if, additionally, N is normal, then M/N is a Lie group.

(iii) Let M, N be Lie groups and let $f\colon M \to N$ be a continuous homomorphism. If f is injective, then $\dim(M) \leq \dim(N)$; if f is surjective, then $\dim(M) \geq \dim(N)$.

(iv) Let M, N be Lie groups with $\dim(M) = \dim(N)$ and with N connected. If $f : M \to N$ is a continuous injective homomorphism, then f is surjective.

Proof. Point (i) is clear. For point (ii) see [7, Theorem 3.42] for the proof that N is Lie and [7, Theorem 3.64] for the proof that M/N is Lie. Point (iii) follows from [7, Theorem 3.32]. As for point (iv), by [7, Theorem 3.32], $f(M)$ is an open, so closed and open, subgroup of N. Since N is connected, $f(M) = N$. □

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Each second countable groups admits a metric generating its topology. If one can find a complete such metric, it is customary to call such a group a *Polish group*. One can check that a second countable group is Polish precisely when it is complete in the sense of [4]. All locally compact second countable groups are Polish. In the proposition below, we collect some basic and well known facts about Polish groups.

Proposition 1.2. Let G be a Polish group.

(i) If there is a continuous homomorphism $f : H \to G$ such that H is a Lie group and $f(H)$ has countable index in G, then G is Lie.

(ii) If H is a closed normal subgroup of G, then G/H with the quotient topology is a Polish group.

(iii) If H is a Polish group and $f : G \to H$ is a Borel measurable homomorphism, then f is continuous. In particular, if K is another Polish group and $f : G \to K$ and $g : H \to K$ are continuous homomorphisms with g being injective and with $f(G) \subseteq g(H)$, then

$$g^{-1} \circ f : G \to H$$

is a continuous homomorphism.

Proof. Point (i) follows from [1, Theorem 2.3.3] after noticing that $f(H)$ is non-meager in G as countably many of its translates cover G. Point (ii) is [1, Theorem 2.2.10]. The first part of point (iii) is a particular case of [1, Theorem 2.3.3]. The second part follows from the first one after observing that g^{-1} is a Borel measurable function. □

2. Theorem

Recall that a second countable group G is *pro-Lie* if it is Polish and each neighborhood of 1 contains a normal subgroup N such that G/N is Lie.

Theorem 2.1. *A closed subgroup of a countable product of Lie groups is pro-Lie.*

Proof. Let L_i, $i \in \mathbb{N}$, be Lie groups, and let $G < \prod_i L_i$ be closed. Let

$$\pi_n \colon \prod_i L_i \to \prod_{i \leq n} L_i \;\text{ and }\; \pi_{n,N} \colon \prod_{i \leq N} L_i \to \prod_{i \leq n} L_i,$$

for $N \geq n$, be projections. The closure in the Lie group $\prod_{i \leq N} L_i$ of the subgroup $\pi_N(G)$ is itself a Lie group, and we let

$$A_N = \text{the connected component of 1 of } \overline{\pi_N(G)}.$$

Let

$$B_{n,N} = \ker\left(\pi_{n,N} \restriction A_N\right).$$

Note that since $\pi_{N,N+1}(A_{N+1})$ is a connected subgroup of $\overline{\pi_N(G)}$, we have

$$\pi_{N,N+1}(A_{N+1}) \subseteq A_N. \tag{1}$$

Claim 1. For every n there is $i_n \geq n$ such that for $N \geq i_n$

$$\dim\left(A_{i_n}/B_{n,i_n}\right) = \dim\left(A_N/B_{n,N}\right).$$

Proof. Let $N \geq n$. Inclusion (1) induces an injective continuous homomorphism

$$A_{N+1}/(\pi^{-1}_{N,N+1}(B_{n,N}) \cap A_{N+1}) \to A_N/B_{n,N}.$$

It follows by Proposition 1.1(iii) that

$$\dim(A_{N+1}/(\pi^{-1}_{N,N+1}(B_{n,N}) \cap A_{N+1})) \leq \dim(A_N/B_{n,N}). \tag{2}$$

Note that

$$\pi^{-1}_{N,N+1}(B_{n,N}) \cap A_{N+1} \subseteq B_{n,N+1},$$

and therefore, by the second part of Proposition 1.1(iii),

$$\dim(A_{N+1}/B_{n,N+1}) \leq \dim(A_{N+1}/(\pi^{-1}_{N,N+1}(B_{n,N}) \cap A_{N+1})).$$

From this inequality and from (2) we get

$$\dim(A_{N+1}/B_{n,N+1}) \leq \dim(A_N/B_{n,N}).$$

We conclude that the natural number valued function $N \to \dim(A_N/B_{n,N})$ is non-increasing, and the claim follows. $\square$

For $n \in \mathbb{N}$, $i_n \geq n$ will denote the natural number from Claim 1.

Claim 2. Let $n \in \mathbb{N}$. For $N \geq i_n$,

$$\pi_{n,N+1}(A_{N+1}) = \pi_{n,N}(A_N).$$

Proof. The homomorphisms $\pi_{n,N} \restriction A_N$ and $\pi_{n,N+1} \restriction A_{N+1}$ induce injective continuous homomorphisms

$$\widehat{\pi}_{n,N} \colon A_N/B_{n,N} \to \prod_{i \leq n} L_i \quad \text{and} \quad \widehat{\pi}_{n,N+1} \colon A_{N+1}/B_{n,N+1} \to \prod_{i \leq n} L_i.$$

Furthermore, from (1), we see that

$$\begin{aligned} \widehat{\pi}_{n,N+1}(A_{N+1}/B_{n,N+1}) &= \pi_{n,N+1}(A_{N+1}) \\ &\subseteq \pi_{n,N}(A_N) = \widehat{\pi}_{n,N}(A_N/B_{n,N}). \end{aligned} \tag{3}$$

By Claim 1 we have

$$\dim(A_{N+1}/B_{n,N+1}) = \dim(A_N/B_{n,N}),$$

and $A_N/B_{n,N}$ is connected, as A_N is. Now since $\widehat{\pi}_{n,N}$ and $\widehat{\pi}_{n,N+1}$ are injective, by (3), we can consider the injective homomorphism

$$(\widehat{\pi}_{n,N})^{-1} \circ \widehat{\pi}_{n,N+1} \colon A_{N+1}/B_{n,N+1} \to A_N/B_{n,N},$$

which is continuous by Proposition 1.2(iii). By what was said above, Proposition 1.1(iv) implies that it is surjective. From this assertion and from (3), the conclusion of the claim follows immediately. □

Claim 3. For every n, for $N \geq i_n$, $\pi_{n,N}(A_N) \subseteq \pi_n(G)$.

Proof. By Claim 2, $\pi_{n,N}(A_N)$ does not depend on N as long as $N \geq i_n$. Put

$$C_n = \pi_{n,N}(A_N),$$

for any $N \geq i_n$.

Fix n_0. We need to see $C_{n_0} \subseteq \pi_{n_0}(G)$. Let y_0 be an arbitrary element of C_{n_0}. Note first that for each n, for $N \geq i_n, i_{n+1}$,

$$C_n = \pi_{n,N}(A_N) = \pi_{n,n+1}(\pi_{n+1,N}(A_N)) = \pi_{n,n+1}(C_{n+1}).$$

It follows from the above equation that there exists $\alpha \in \prod_i L_i$ such that $\pi_{n_0}(\alpha) = y_0$, and for every $m \geq n_0$, $\pi_m(\alpha) \in C_m$. Now from the definitions of A_m and C_m we see that for each $m \geq n_0$ we have, with arbitrary $N \geq i_m$,

$$\pi_m(\alpha) \in \pi_{m,N}\left(\overline{\pi_N(G)}\right) \subseteq \overline{\pi_m(G)},$$

This allows us to pick a sequence $g^m \in G$ such that for each i we have in L_i

$$g^m(i) \to \alpha(i) \;\text{ as } m \to \infty. \tag{4}$$

Since G is closed, it follows that $\alpha \in G$. □

Let for $n \in \mathbb{N}$,

$$L_{(n)} = \{1\}^n \times \prod_{i>n} L_i.$$

We are ready to finish the proof by showing that for every n, $G/\left(G \cap L_{(n)}\right)$ is Lie. Fix n and note that $G/\left(G \cap L_{(n)}\right)$ is a Polish group by Proposition 1.2(ii). Thus, by Proposition 1.2(i), it suffices to find in it a subgroup of countable index that is a continuous injective image of a Lie group.

Take $N \geq i_n$. Consider the following commutative diagram. The functions in the diagram are defined below it.

$$\begin{tikzcd}
A_N \arrow[rr, "\pi_{n,N}\restriction A_N"] \arrow[drr, "\sigma"'] & & \pi_n(G) \\
& & G/\left(G \cap L_{(n)}\right) \arrow[u, "\rho"']
\end{tikzcd}$$

The range of the continuous homomorphism $\pi_{n,N} \restriction A_N$ is included in the group $\pi_n(G)$ by Claim 3. The function ρ is the composition of the natural continuous injective homomorphism $G/\left(G \cap L_{(n)}\right) \to (\prod_i L_i)/L_{(n)}$ and the natural continuous isomorphism $(\prod_i L_i)/L_{(n)} \to \prod_{i \leq n} L_i$. Clearly ρ is a continuous isomorphism. We define σ by

$$\sigma = \rho^{-1} \circ \pi_{n,N}.$$

By Proposition 1.2(iii), σ is a continuous homomorphism.

Since A_N is Lie, it suffices to show that $\sigma(A_N)$ has countable index in the group $G/\left(G \cap L_{(n)}\right)$. Since ρ is an isomorphism, it follows from the diagram that it is enough to prove that $\pi_{n,N}(A_N)$ has countable index in $\pi_n(G)$. To see this note that from the definition of A_N and from Proposition 1.1(i), A_N is a non-empty relatively open subset of $\overline{\pi_N(G)}$, which allows us to pick $g_j \in G$, $j \in \mathbb{N}$, so that

$$\bigcup_j \pi_N(g_j) A_N = \overline{\pi_N(G)}.$$

Applying $\pi_{n,N}$ to both sides of the equality and noticing that $\pi_{n,N}(\overline{\pi_N(G)}) \supseteq \pi_n(G)$, we see that countably many translates of $\pi_{n,N}(A_N)$ cover $\pi_n(G)$ as required. $\square$

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