Fiber Bundles with Group Actions

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We extend some of the most important results on the construction and characterization of universal bundles from the usual theory of bundles to the setting of fiber bundles with a group action. In particular, this includes the "real" vector bundles in the sense of Atiyah [Ati66]. To conclude, we give an application to characteristic classes.

1 Definition of the Bundles

1.1 Let Γ be a compact Lie group, G a topological group, and $\alpha : \Gamma \to \mathrm{Aut}(G)$ a homomorphism from Γ to the automorphism group of G. The image of γ under α will be denoted by α_γ. We require the map $\Gamma \times G \to G$ given by $(\gamma, g) \mapsto \alpha_\gamma(g)$ to be continuous. In what follows, topological spaces and groups will always be Hausdorff spaces.

Definition. A (Γ, α, G)-bundle consists of a locally trivial principal G-bundle $p : E \to B$ and a continuous left action of Γ on E and B such that the following holds:

(A1) The map p is equivariant, i.e., for all $e \in E$ and $\gamma \in \Gamma$, we have $p(\gamma e) = \gamma p(e)$.

(A2) For all $\gamma \in \Gamma$, $g \in G$, and $e \in E$, we have $\gamma(eg) = (\gamma e)\alpha_\gamma(g)$.

(As usual, we assume that G acts on E from the right.) We often refer to a (Γ, α, G)-bundle only by the underlying map $p : E \to B$. If α is trivial, then Γ acts by bundle automorphisms; this is a Γ-equivariant bundle in the actual sense (cf. [AS65]).

A map between (Γ, α, G)-bundles is a bundle map of the underlying principal G-bundles which is compatible with the action of Γ. A Γ-map $f : C \to B$ induces a (Γ, α, G)-bundle over C from one over B. A bundle map is a equivalence in the category of (Γ, α, G)-bundles if it induces a homeomorphism on the base space.

Associated fiber bundles: if F is a space on which Γ and G act from the left, in such a way that $\gamma(gf) = \alpha_\gamma(g)(\gamma f)$, the bundle associated to $p : E \to B$ with fiber F has a canonical Γ-action.

1.2 The data Γ, α, and G produce a semidirect product $\Gamma \ltimes_\alpha G$[1] in the following way: on the topological product $\Gamma \times G$ we define an operation via

$$(\gamma, g)(\gamma, g') = (\gamma\gamma', \alpha_\gamma g' \cdot g).$$

*Translated from the German *Faserbündel mit Gruppenoperation* by Chase Vogeli on September 18, 2022. Any mistakes are assumed to be introduced in translation.

[1] We use the more modern notation of $\ltimes_\alpha$ for semidirect products as opposed to $\times_\alpha$ as it appears in the original.

Under the given conditions on α, we obtain a topological group in this way which we denote by $\Gamma \ltimes_\alpha G$. The group $\Gamma \ltimes_\alpha G$ acts from the left on the total space E of a (Γ, α, G)-bundle via $(\gamma, g)e = (\gamma e)g$. Conversely, every $\Gamma \ltimes_\alpha G$-space E produces a space with actions of Γ and G such that (A2) holds.

1.3. Examples Let $\alpha : \mathbb{Z}_2 \to \mathrm{Aut}(U(n))$ be the homomorphism that associates to the nontrivial element of $\mathbb{Z}_2$ the transformation from a unitary matrix $g \in U(n)$ to its complex conjugate $\overline{g}$. Then, $(\mathbb{Z}_2, \alpha, U(n))$-bundles are the principal bundles associated to the real vector bundles over real spaces ("real" in the sense of [Ati66]). Or:

$$\alpha' : \Gamma \xrightarrow{\beta} \mathbb{Z}_2 \xrightarrow{\alpha} \mathrm{Aut}(U(n))$$

for β continuous and α as above.

2 Local Objects

2.1 Let Λ be a closed subgroup of Γ. The space Γ/Λ of cosets $\gamma\Lambda$ is a Γ-space via left translation.

Definition. A local object is a (Γ, α, G)-bundle over Γ/Λ.

We seek to classify local objects. Let $p : E \to \Gamma/\Lambda$ be a local object and $x \in E$ a point with $px = \Lambda$. Let $H_x \subset \Gamma \ltimes_\alpha G$ be the stabilizer of the point x. The group H_x is closed, as E is a Hausdorff space. We regard H_x as a subspace of $\Gamma \times G$. If $(\gamma, g) \in H_x$, then $\gamma \in \Lambda$, so from $(\gamma x)g = x$ it follows that $\Lambda = px = p((\gamma x)g) = \gamma\Lambda$. If (γ, g) and (γ, h) are in H_x, then $g = h$, since G acts freely on E. Therefore, H_x is the graph of a map $t : \Gamma \to G$; we have $t(\lambda\lambda') = \alpha_\lambda(t\lambda') \cdot t\lambda$. If α is trivial, t is an antihomomorphism. Furthermore, t is continuous, since p is assumed to be a locally trivial principal G-bundle. The map $f : \Gamma \times_\alpha G/H_x \to E$ given by $u \mapsto ux$ is compatible with the action of $\Gamma \ltimes_\alpha G$. With the help of the following lemma, we see that f is a homeomorphism.

Lemma. *Let the closed subgroup H of $\Gamma \ltimes_\alpha G$ be the graph of a continuous map $t : \Lambda \to G$. Then,*

$$r : \Gamma \times_\alpha G/H \to \Gamma/\Lambda,$$

regarded as a principal G-bundle, is locally trivial.

Proof. The fact that $q : \Gamma \to \Gamma/\Lambda$ admits local sections is crucial. Let $s : U \to \Gamma$ be a section of q on an open set $U \subset \Gamma/\Lambda$. We consider the diagram

$$\begin{tikzcd} q^{-1}U \times G \arrow[r, shift right, "\Psi"'] \arrow[d, "a"] & U \times \Lambda \times G \arrow[l, shift right, "\Phi"'] \arrow[d, "b"] \\ r^{-1}U & U \times G \arrow[l, "\varphi"] \end{tikzcd}$$

with the maps: $b(u, \lambda, g) = (u, g)$, $\varphi(u, g) = a(su, g)$, a the restriction of $\Gamma \times G \to \Gamma \times_\alpha G/H$, and

$$\Phi(u, \lambda, g) = (su \cdot \lambda, \alpha_{s(u)}(t\lambda) \cdot g),$$
$$\Psi(x, y) = (qx, (sqx)^{-1}x, \alpha_{sqx}^{-1} t((sqx)^{-1}x) \cdot y).$$

Then, $a\Phi = \varphi b$. Furthermore, $\Phi\Psi = \mathrm{id}$ and $\Psi\Phi = \mathrm{id}$. The map φ is thus a homeomorphism over U which is compatible with the right action of G. $\square$

2.2 If H and K are two subgroups of $\Gamma \ltimes_\alpha G$ regarded as above, there exists a bundle map of the corresponding (Γ, α, G)-bundles (over a Γ-isomorphism from Γ/Λ) exactly when H and K are conjugate in $\Gamma \times \alpha G$. We call such objects isomorphic.

Theorem. *Let G be a Lie group with finitely many components. Then, there exists only countably many local objects up to isomorphism (for a fixed Γ, α, G).*

Proof. The group $\Gamma \ltimes_\alpha G$ is a Lie group. It has only countably many conjugacy classes of compact subgroups ([Pal60, p. 1.7.27]; in conjunction with [Hoc65, Theorem 1.3, §180]). By 2.1 and the remark before the theorem, only such subgroups need to be regarded. $\square$

Remark. In the preceding theorem, we could have more generally assumed G to be the limit of a sequence $G_1 \subset G_2 \subset \cdots$ of Lie groups with finitely many components.

2.3. Definition A (Γ, α, G)-bundle $p : E \to B$ is locally trivial, when B has an open cover $U = (U_j | j \in J)$ with the following properties:

(T1) The U_j are Γ-invariant (as sets).

(T2) The restriction $p^{-1}U_j \to U_j$ admits a (Γ, α, G)-bundle map to a local object.

The local object in (T2) is not uniquely determined. A locally trivial (Γ, α, G)-bundle is said to be numerable, when U has an subordinate locally finite partition of unity. If Γ is compact, we are able to integrate over the group and assume that the functions in the partition of unity are Γ-invariant.

Theorem. *Let G be a compact Lie group. Let $p : E \to B$ be a (Γ, α, G)-bundle with completely regular total space E. Then, the bundle is locally trivial in sense defined above.*

Proof. Let $x \in E$ be given. Then, there exists a $(\Gamma \ltimes_\alpha G)$-neighborhood U of x and a equivariant map $f : U \to (\Gamma \ltimes_\alpha G)/H_x$. We have this from [Pal60, pp. 1.7.7, 1.7.19]. The map f provides the desired bundle map to a local object. $\square$

3 Construction of Universal Bundles

3.1 Let Γ, α, and G be given as above. Let $p_j : X_j \to Y_j$ $j \in J$ be a family of local objects such that every local object is isomorphic (cf. 2.2) to one in this family. Let $E_j = X_j * X_j * \cdots$ be the join of countably infinitely many instances of X_j. As is customary, the topology on E_j is the coarsest for which all "coordinate maps" are continuous (Milnor [Mil56]). We further form the join

$$E(\Gamma, \alpha, G) = *_{j \in J} E_j$$

of the E_j, with the aforementioned topology. The spaces $E(\Gamma, \alpha, G)$ and E_j are $\Gamma \ltimes_\alpha G$-spaces in a canonical way. We set $B(\Gamma, \alpha, G) = E(\Gamma, \alpha, G)/G$ and obtain a Γ-map from the quotient map $p : E(\Gamma, \alpha, G) \to B(\Gamma, \alpha, G)$.

Theorem. *(a) p is a locally trivial (Γ, α, G)-bundle.*

(b) Every numerable (Γ, α, G)-bundle admits a bundle map to p. Any two such bundle maps are homotopic as (Γ, α, G)-bundle maps.

(c) If J is countable, then p is numerable (cf. 2.3).

Proof. (a) (cf. [Mil56]) There are coordinate functions $t_{j,n} : E(\Gamma, \alpha, G) \to [0, 1]$ for $j \in J$, $n = 1, 2, \ldots$ (the nth t-coordinate from E_j). Let $V_{j,n}$ be the preimage of $[0, 1]$ under $t_{j,n}$. The bundle $V_{j,n} \to V_{j,n}/G$ admits a bundle map to p_j. The subspace $V_{j,n}$ is open and Γ-invariant in $B(\Gamma, \alpha, G)$.

(b) The existence of a bundle map is shown as in [Die66, p. 5]. (The definition of "locally trivial" in 2.3 is adapted from there.) In order to construct a homotopy between two bundle maps, the proof method from [Die66, p. 6] can be applied to each E_j separately. The homotopy is compatible with the (Γ, α, G)-structure.

(c) See [Dol63, p. 251]. □

3.2 Henceforth let $E = E(\Gamma, \alpha, G)$ and $q : X \to \Gamma/\Lambda$ be a local object. The fibration $f_q : (X \times E)/G \to \Gamma/\Lambda$ associated to q with fiber E is a Γ-map. We say f_q is Γ-shrinkable (cf. [Dol63, p. 225]) if there exists a Γ-section $s : \Gamma/\Lambda \to (X \times E)/G$ of f_q such that sf_q is Γ-homotopic over Γ/Λ (i.e. fiberwise) to the identity of $(X \times E)/G$.

Theorem. *For any local object q, f_q is Γ-shrinkable.*

Proof. We describe points in E via families $(a_{j,n} | j \in J, n = 1, 2, \ldots)$ in which $a_{j,n} = t_{j,n}x_{j,n}$, $t_{j,n} \in I$, and $x_{j,n} \in X_j$; furthermore only finitely many t_j, n are nonzero and $\sum t_{j,n} = 1$. Let $q : X \to \Gamma/\Lambda$ be isomorphic to $X_0 \to Y_0$, $0 \in J$. We assume that $X = X_0$. The space X is of the form $(\Gamma \ltimes_\alpha G)/H$. Therefore, we describe points from X by their representatives in $\Gamma \times G$.

We define a section s as

$$s(\gamma\Lambda) = ((\gamma, e), a_{j,n}(\gamma)),$$

in which $a_{0,1}(\gamma) = 1(\gamma, e)$ and $a_{j,n}(\gamma) = 0(\gamma, e)$ for $(j, n) \neq (0, 1)$, with $e \in G$ being the identity element. The map s is well-defined, a Γ-map, and a section.

We construct the desired Γ-homotopy from sf to the identity as follows. Let $H_t : (X \times E)/G \to (X \times E)/G$ be defined as

$$a_{j,n} = \begin{cases} tt_{j,n}x_{j,n} & j \neq 0, \\ tt_{0,n-1}x_{0,n-1} & j = 0, n > 1, \\ (1-t)(\gamma, g) & j = 0, n = 1. \end{cases}$$

Then, $H_0 = df$ and H_1 is almost equal to the identity, namely in the coordinates with $j \neq 0$. Similarly to in [Die66, p. 6], we now construct a homotopy in infinitely many steps which connects $(0x_1, t_1x_2, t_2x_2, \ldots) \in E_0$ to $(t_1x_1, t_2x_2, \ldots) \in E_0$ and provides a homotopy between H_1 and the identity on $(X \times E)/G$. One checks that these homotopies are compatible with the action of Γ and lie over Γ/Λ. □

3.3 In this manner one can construct universal bundles for restricted categories of (Γ, α, G)-bundles. For example: trivial Γ-action on the base space of the bundle; or the base space has finitely many orbits; or the Γ-action on the base space factors through a quotient group of Γ. Of course the join in general does not "contain" only the local objects from which it is formed. Therefore the reduced join (in the sense of [Pal60, p. 1.3.6]) is to be used.

In the construction of universal bundles, one also does not have to start from all isomorphism types of local objects. One may restrict to a set of local objects, such that any local object maps to a one in the set.

4 The Homotopy Theorem

Let $p : X \to B \times I$ be a numerable (Γ, α, G)-bundle. Let $r : B \times I \to B \times I$ be the map $r(b, t) = (b, 1)$.

Theorem. *Under the following conditions there exists a* (Γ, α, G)*-bundle map* $p \to p$ *lying over* r*:*

(a) All of the orbits of B *consist of finitely many points (e.g.,* Γ *is a finite group).*

(b) The space B *is the limit of a sequence* $B_1 \subset B_2 \subset \cdots$ *of compact* Γ*-spaces* B_i *and* G *is a compact Lie group.*

Consequence. Let $p : X \to C$ be a (Γ, α, G)-bundle and $H : B \times I \to C$ be a Γ-homotopy (i.e., H_t is a Γ-map for all $t \in I$). Let the bundle $H^*(p)$ induced from p by H be numerable (e.g., if p is numerable). Then, under either conditions (a) or (b) of the theorem, the bundle $H^*(p)$ is equivalent to the product bundle $H_0^*(p) \times I$. In particular, $H_0^*(p)$ and $H_1^*(p)$ are equivalent. □

Proof of the theorem. (a) In this case the proof in [Hus66, p. 48-50] transfers nearly verbatim. This is essentially because homotopies in the base space of the local objects in question are constant. If we say a bundle is trivial when it admits a map to a local object, one only needs to consider a slightly different proof from [Hus66, Lemma 4.9.4, p. 49].

(b) First, let B be compact. Let $r_t : B \times I \to B \times I$ be given by $r_t(b, s) = (b, t)$, so $r = r_1$. Let $g_t : B \times I \to (B \times I) \times (B \times I)$ be given by $g_t(u) = (u, r_t u)$ and let $q_t : Y_t \to B \times I$ be the map induced by the product $p \times p$ through g_t. The map q_t is a $(\Gamma, \alpha', G \times G)$-bundle in a canonical way, where $\alpha' : \Gamma \to \mathrm{Aut}(G \times G)$ is given by $\alpha'_\gamma(g, h) = (\alpha_\gamma(g), \alpha_\gamma(h))$. Let $K = \Gamma \times_{\alpha'} (G \times G)$. There is a right $G \times G$-action on G via $x(g, h) = g^{-1}xh$ and a left Γ-action via $\gamma x = \alpha_\gamma(x)$. We have

$$\gamma(x(g, h)) = (\gamma x) \cdot \alpha'_\gamma(g, h).$$

In this way G becomes a K-space. Bundle maps $p \to p$ over r_t correspond to Γ-sections of q_t and these to K-maps $f : Y_t \to G$ (cf. [Hus66, 4.8, p.46-47]). Let $Y(t)$ be the part of Y_t lying over $B \times \{t\}$. We have a K-map $\alpha : Y(t) \to G$. It admits an extension from a neighborhood $Y(t)$ to Y_t [Pal60, p. 1.6]. From the compactness of B, there is a neighborhood $[a_t, b_t]$ for t in I, such that a bundle map from p restricted to $p^{-1}(B \times [a_t, b_t])$, over r_t restricted to $B \times [a_t, b_t]$, exists, which is the identity over $B \times \{t\}$. Thus, $p^{-1}(B \times [a_t, b_t])$ is K-isomorphic over $B \times [a_t, b_t]$ to $p^{-1}(B \times \{t\}) \times [a_t, b_t]$. The claim follows easily.

For a sequence (B_t) of compact Γ-spaces, one can construct the bundle map with the previously proven one inductively over $B_i \times I \cup \lim B_i \times 0$. We do not go any further in detail. □

5 Characterization of Universal Bundles

5.1. Theorem. *Let $p : E \to B$ be a (Γ, α, G)-bundle, such that for every local object $q : X \to \Gamma/\Lambda$, the associated bundle $(X \times E)/G \to \Gamma/\Lambda$ is Γ-shrinkable. Then, every numerable (Γ, α, G)-bundle admits a bundle map to p and every two such bundle maps are homotopic.*

Proof. Let $V \to Y$ be a numerable (Γ, α, G)-bundle. We consider the associated bundle $(V \times E)/G \to Y$.

Claim: this associated bundle has the Γ-section extension property (Γ-SEP). By [Dol63, p. 2.8], it suffices to show that the local Γ-SEP holds. Locally, the bundle is induced by Γ-shrinkable bundles, which by [Dol63, p. 3.1] have the Γ-SEP. The rest of the proof carries on as in [Dol63, p. 249]. □

Theorem. *Let Γ be finite and G the limit of a sequence $G_1 \subset G_2 \subset \cdots$ of Lie groups with finitely many components. A numerable (Γ, α, G)-bundle $E \to B$ is universal, if for every local object $X \to \Gamma/\Lambda$, the associated bundle $(X \times E)/G \to \Gamma/\Lambda$ is Γ-shrinkable.*

Proof. By 2.2, 3.1, and 3.2, there are numerable universal (Γ, α, G)-bundles, for which all bundles $(X \times E)/G \to \Gamma/\Lambda$ are Γ-shrinkable. With 4 and 5.1, the proof is continued as in [Dol63, Theorem 7.5, p. 249]. □

Remark. "Universal" here indicates the validity of the homotopy theorem.

5.3. Example. For $(\mathbb{Z}_2, \alpha, U(n))$-bundles (cf. 1.3) there are two local objects up to isomorphism. With the notation of 2.1: $\Lambda = 0$ and t trivial; $\Lambda = \mathbb{Z}_2$ and t trivial. For the construction of universal bundles one can reduce to the case of $U(n)$ over a point with the $\mathbb{Z}_2$-action given by conjugation (by the concluding remark in 3.3). The classifying space for these bundles is thus $BU(n)$ with the involution induced by conjugation.

6 Characteristic Classes

The universal principal G-bundle $EG \to BG$ from Milnor [Mil56] admits a Γ-action via

$$\gamma \cdot (t_1 g_1, t_2 g_2, \ldots) = (t_1 \alpha_\gamma(g_1), t_2 \alpha_\gamma(g_2), \ldots).$$

We denote the (Γ, α, G)-bundle given as such by $p_0 : E_0 \to B_0$. Let $j : B_0 \to B(\Gamma, \alpha, G)$ be a classifying map for p_0. Let $J = \mathrm{id} \times_\Gamma j : E\Gamma \times_\Gamma B_0 \to E\Gamma \times_\Gamma B(\Gamma, \alpha, G)$. We assume, that the universal (Γ, α, G)-bundle is numerable (cf. 3.1).

Theorem. *J and j (after forgetting the Γ-structure) are homotopy equivalences.*

Proof. It follows from 3.2 that $E(\Gamma, \alpha, G)$ is contractible. Thus, p is a universal principal G-bundle by forgetting the Γ-structure. Since j induces a universal principal G-bundle from another, it must be that j is a homotopy equivalence. The claim about J now follows from [Dol63, Theorem 6.3]. □

Let α be trivial and be suppressed from the notation. We begin with a numerable (Γ, G)-bundle $p : V \to X$ over a free Γ-space X such that the quotient bundle $q : X \to X/\Gamma$ is numerable. Let $k : X \to B(\Gamma, G)$, $k' : X/\Gamma \to BG$, and $(L, l) : (X, X/\Gamma) \to (E\Gamma, B\Gamma)$ be classifying maps from p, $p/\Gamma : V/\Gamma \to X/\Gamma$, and q. The associated fibration $r : E\Gamma \times_\Gamma X \to X/\Gamma$ is a homotopy equivalence, since the fibers are contractible. A inverse equivalence $s : X/\Gamma \to E\Gamma \times_\Gamma X$ is provided by $s(\Gamma x) = (Lx, x) \mod \Gamma$.

Theorem. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} E\Gamma \times_\Gamma X & \xrightarrow{\mathrm{id} \times_\Gamma k} & E\Gamma \times_\Gamma B(\Gamma, G) \\ \downarrow r & & \uparrow J \\ X/\Gamma & \xrightarrow{[I, k']} & B\Gamma \times BG \end{array}$$

Proof. First of all, $jk'q$ is Γ-homotopic to k, since both maps induce the same (Γ, G)-bundle over X. Thus, $\mathrm{id} \times_\Gamma jk'q$ is homotopic to $\mathrm{id} \times_\Gamma k$. We assemble $\mathrm{id} \times_\Gamma k'q$ with q to obtain $[l, k']$. Since r and s are inverse homotopy equivalences, the claim follows. $\square$

6.3 If k is a functor defined on the category of topological space, we obtain a functor ${}_\Gamma k$ defined on the category of Γ-spaces from it by defining

$${}_\Gamma k(X) = k(E\Gamma \times_\Gamma X).$$

We seek characteristic classes for $(\Gamma, O(n))$-bundles with values in cohomology ${}_\Gamma H^*$. The universal classes are thus elements in ${}_\Gamma H^*(B(\Gamma, O(n)))$, or—due to the homotopy equivalence from 6.1—elements in $H^*(B\Gamma \times BO(n))$.

Characteristic Classes for bundles over free Γ-spaces were considered by Conner–Floyd (e.g., in [CF64, p. 91]). The connection between there and the classes considered here is the theorem from 6.2, which gives a splitting of the classes that goes beyond that of 6.1 For instance, using cohomology with coefficients in a field L, so that we have the Künneth formula

$$H^*(B\Gamma \times BO(n); L) \cong H^*(B\Gamma; L) \otimes H^*(BO(n); L)$$

at our disposal, all characteristic classes from $V \to X$ are polynomial in characteristic classes from $X \to X/\Gamma$ and $V/\Gamma \to X/\Gamma$.

Corrections (March 3, 1969). The homotopy Theorem 4 (and also Theorem 5.2) apply generally for compact Lie groups Γ. The proof of 4 (a) can be carried out "inductively over Γ," if one considers the following: Let $p : X \to B \times [0, 1]$ be induced by a local object through $f : B \times [0, 1] \to \Gamma/\Lambda$. Then, B is of the form $\Gamma \times_\Lambda Y$ with $Y = f_0^{-1}(\Lambda)$ [Pal60, p. 1.7.9]. The restriction p' of p to $Y \times [0, 1]$ is equivalent to $p'_0 \times \mathrm{id}$ (this is clear for $\Gamma = \Lambda$ and otherwise via induction [Pal60, p. 1.8.1]). Then. $p = \Gamma \times_\Lambda p' = (\Gamma \times p'_0) \times \mathrm{id}$. The theorem applies also locally. One completes the proof as usual.

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