Fiber Bundles with Group Actions

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We extend some of the most important results on the construction and characterization of universal bundles from the usual theory of bundles to the setting of fiber bundles with a group action. In particular, this includes the “real” vector bundles in the sense of Atiyah [Ati66]. To conclude, we give an application to characteristic classes.

1 Definition of the Bundles

1.1 Let \( \Gamma \) be a compact Lie group, \( G \) a topological group, and \( \alpha: \Gamma \to \text{Aut}(G) \) a homomorphism from \( \Gamma \) to the automorphism group of \( G \). The image of \( \gamma \) under \( \alpha \) will be denoted by \( \alpha_\gamma \). We require the map \( \Gamma \times G \to G \) given by \( (\gamma, g) \mapsto \alpha_\gamma(g) \) to be continuous. In what follows, topological spaces and groups will always be Hausdorff spaces.

**Definition.** A \((\Gamma, \alpha, G)\)-bundle consists of a locally trivial principal \( G \)-bundle \( p: E \to B \) and a continuous left action of \( \Gamma \) on \( E \) and \( B \) such that the following holds:

A1) The map \( p \) is equivariant, i.e., for all \( e \in E \) and \( \gamma \in \Gamma \), we have \( p(\gamma e) = \gamma p(e) \).

A2) For all \( \gamma \in \Gamma \), \( g \in G \), and \( e \in E \), we have \( \gamma(eg) = (\gamma e)\alpha_\gamma(g) \).

(As usual, we assume that \( G \) acts on \( E \) from the right.) We often refer to a \((\Gamma, \alpha, G)\)-bundle only by the underlying principal bundle \( p: E \to B \). If \( \alpha \) is trivial, then \( \Gamma \) acts by bundle automorphisms; this is a \( \Gamma \)-equivariant bundle in the actual sense (cf. [AS65]).

A map between \((\Gamma, \alpha, G)\)-bundles is a bundle map of the underlying principal \( G \)-bundles which is compatible with the action of \( \Gamma \). A \( \Gamma \)-map \( f: C \to B \) induces a \((\Gamma, \alpha, G)\)-bundle over \( C \) from one over \( B \). A bundle map is a equivalence in the category of \((\Gamma, \alpha, G)\)-bundles if it induces a homeomorphism on the base space.

Associated fiber bundles: if \( F \) is a space on which \( \Gamma \) and \( G \) act from the left, in such a way that \( \gamma(gf) = \alpha_\gamma(g)(\gamma f) \), the bundle associated to \( p: E \to B \) with fiber \( F \) has a canonical \( \Gamma \)-action.

1.2 The data \( \Gamma \), \( \alpha \), and \( G \) produce a semidirect product \( \Gamma \ltimes_\alpha G \) in the following way: on the topological product \( \Gamma \times G \) we define an operation via

\[
(\gamma, g)(\gamma', g') = (\gamma \gamma', \alpha_\gamma g' \cdot g).
\]

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1We use the more modern notation of \( \ltimes_\alpha \) for semidirect products as opposed to \( \times_\alpha \) as it appears in the original.
Under the given conditions on $\alpha$, we obtain a topological group in this way which we denote by $\Gamma \ltimes_\alpha G$. The group $\Gamma \ltimes_\alpha G$ acts from the left on the total space $E$ of a $(\Gamma, \alpha, G)$-bundle via $(\gamma, g)e = (\gamma e)g$. Conversely, every $\Gamma \ltimes_\alpha G$-space $E$ produces a space with actions of $\Gamma$ and $G$ such that (A2) holds.

1.3. Examples  Let $\alpha : \mathbb{Z}_2 \to \text{Aut}(U(n))$ be the homomorphism that associates to the nontrivial element of $\mathbb{Z}_2$ the transformation from a unitary matrix $g \in U(n)$ to its complex conjugate $g^*$. Then, $(\mathbb{Z}_2, \alpha, U(n))$-bundles are the principal bundles associated to the real vector bundles over real spaces (“real” in the sense of [Ati66]). Or:

$$\alpha' : \Gamma \xrightarrow{\beta} \mathbb{Z}_2 \xrightarrow{\alpha} \text{Aut}(U(n))$$

for $\beta$ continuous and $\alpha$ as above.

2 Local Objects

2.1 Let $\Lambda$ be a closed subgroup of $\Gamma$. The space $\Gamma/\Lambda$ of cosets $\gamma\Lambda$ is a $\Gamma$-space via left translation.

Definition. A local object is a $(\Gamma, \alpha, G)$-bundle over $\Gamma/\Lambda$.

We seek to classify local objects. Let $p : E \to \Gamma/\Lambda$ be a local object and $x \in E$ a point with $px = \Lambda$. Let $H_x \subset \Gamma \ltimes_\alpha G$ be the stabilizer of the point $x$. The group $H_x$ is closed, as $E$ is a Hausdorff space. We regard $H_x$ as a subspace of $\Gamma \times G$. If $(\gamma, g) \in H_x$, then $\gamma \in \Lambda$, so from $(\gamma x)g = x$ it follows that $\Lambda = px = p((\gamma x)g) = \gamma \Lambda$. If $(\gamma, g)$ and $(\gamma, h)$ are in $H_x$, then $g = h$, since $G$ acts freely on $E$. Therefore, $H_x$ is the graph of a map $t : \Gamma \to G$; we have $t(\lambda \lambda') = \alpha_\lambda(t\lambda') \cdot t\lambda$. If $\alpha$ is trivial, $t$ is an antihomomorphism. Furthermore, $t$ is continuous, since $p$ is assumed to be a locally trivial principal $G$-bundle. The map $f : \Gamma \times_\alpha G/H_x \to E$ given by $u \mapsto ux$ is compatible with the action of $\Gamma \ltimes_\alpha G$. With the help of the following lemma, we see that $f$ is a homeomorphism.

Lemma. Let the closed subgroup $H$ of $\Gamma \ltimes_\alpha G$ be the graph of a continuous map $t : \Lambda \to G$. Then,

$$r : \Gamma \times_\alpha G/H \to \Gamma/\Lambda,$$

regarded as a principal $G$-bundle, is locally trivial.

Proof. The fact that $q : \Gamma \to \Gamma/\Lambda$ admits local sections is crucial. Let $s : U \to \Gamma$ be a section of $q$ on an open set $U \subset \Gamma/\Lambda$. We consider the diagram

$$
\begin{array}{ccc}
q^{-1}U \times G & \xleftarrow{\Phi} & U \times \Lambda \times G \\
\downarrow a & & \downarrow b \\
r^{-1}U & \xleftarrow{\varphi} & U \times G \\
\end{array}
$$

with the maps: $b(u, \lambda, g) = (u, g)$, $\varphi(u, g) = a(su, g)$, $a$ the restriction of $\Gamma \times G \to \Gamma \times_\alpha G/H$, and

$$
\Phi(u, \lambda, g) = (su \cdot \lambda, \alpha_{s(u)}(t\lambda) \cdot g),
\Psi(x, y) = (qx, (sqx)^{-1}x, \alpha_{sqx}^{-1}t((sqx)^{-1}x) \cdot y).
$$

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Then, $a\Phi = \varphi b$. Furthermore, $\Phi \Psi = \text{id}$ and $\Psi \Phi = \text{id}$. The map $\varphi$ is thus a homeomorphism over $U$ which is compatible with the right action of $G$. \hfill \Box

2.2 If $H$ and $K$ are two subgroups of $\Gamma \ltimes \alpha G$ regarded as above, there exists a bundle map of the corresponding $(\Gamma, \alpha, G)$-bundles (over a $\Gamma$-isomorphism from $\Gamma/\Lambda$) exactly when $H$ and $K$ are conjugate in $\Gamma \times \alpha G$. We call such objects isomorphic.

**Theorem.** Let $G$ be a Lie group with finitely many components. Then, there exists only countably many local objects up to isomorphism (for a fixed $\Gamma$, $\alpha$, $G$).

**Proof.** The group $\Gamma \ltimes \alpha G$ is a Lie group. It has only countably many conjugacy classes of compact subgroups ([Pal60, p. 1.7.27]; in conjunction with [Hoc65, Theorem 1.3, §180]). By 2.1 and the remark before the theorem, only such subgroups need to be regarded. \hfill \Box

**Remark.** In the preceding theorem, we could have more generally assumed $G$ to be the limit of a sequence $G_1 \subset G_2 \subset \cdots$ of Lie groups with finitely many components.

2.3. **Definition** A $(\Gamma, \alpha, G)$-bundle $p : E \to B$ is locally trivial, when $B$ has an open cover $U = (U_j | j \in J)$ with the following properties:

(T1) The $U_j$ are $\Gamma$-invariant (as sets).

(T2) The restriction $p^{-1}U_j \to U_j$ admits a $(\Gamma, \alpha, G)$-bundle map to a local object.

The local object in (T2) is not uniquely determined. A locally trivial $(\Gamma, \alpha, G)$-bundle is said to be numerable, when $U$ has an subordinate locally finite partition of unity. If $\Gamma$ is compact, we are able to integrate over the group and assume that the functions in the partition of unity are $\Gamma$-invariant.

**Theorem.** Let $G$ be a compact Lie group. Let $p : E \to B$ be a $(\Gamma, \alpha, G)$-bundle with completely regular total space $E$. Then, the bundle is locally trivial in sense defined above.

**Proof.** Let $x \in E$ be given. Then, there exists a $(\Gamma \ltimes \alpha G)$-neighborhood $U$ of $x$ and an equivariant map $f : U \to (\Gamma \ltimes \alpha G)/H_x$. We have this from [Pal60] pp. 1.7.7, 1.7.19. The map $f$ provides the desired bundle map to a local object. \hfill \Box

3 **Construction of Universal Bundles**

3.1 Let $\Gamma$, $\alpha$, and $G$ be given as above. Let $p_j : X_j \to Y_j$, $j \in J$ be a family of local objects such that every local object is isomorphic (cf. 2.2) to one in this family. Let $E_j = X_j \star X_j \star \cdots$ be the join of countably infinitely many instances of $X_j$. As is customary, the topology on $E_j$ is the coarsest for which all “coordinate maps” are continuous (Milnor [Mil56]). We further form the join

$$E(\Gamma, \alpha, G) = \star_{j \in J} E_j$$

of the $E_j$, with the aforementioned topology. The spaces $E(\Gamma, \alpha, G)$ and $E_j$ are $\Gamma \ltimes \alpha G$-spaces in a canonical way. We set $B(\Gamma, \alpha, G) = E(\Gamma, \alpha, G)/G$ and obtain a $\Gamma$-map from the quotient map $p : E(\Gamma, \alpha, G) \to B(\Gamma, \alpha, G)$. 

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Theorem. (a) \( p \) is a locally trivial \((\Gamma, \alpha, G)\)-bundle.
(b) Every numerable \((\Gamma, \alpha, G)\)-bundle admits a bundle map to \( p \). Any two such bundle maps are homotopic as \((\Gamma, \alpha, G)\)-bundle maps.
(c) If \( J \) is countable, then \( p \) is numerable (cf. 2.3).

Proof. (a) (cf. [Mil56]) There are coordinate functions \( t_{j,n} : E(\Gamma, \alpha, G) \to [0, 1] \) for \( j \in J, n = 1, 2, \ldots \) (the \( n \)th \( t \)-coordinate from \( E \)). Let \( V_{j,n} \) be the preimage of \([0, 1] \) under \( t_{j,n} \). The bundle \( V_{j,n} \to V_{j,n}/G \) admits a bundle map to \( p_j \). The subspace \( V_{j,n} \) is open and \( \Gamma \)-invariant in \( B(\Gamma, \alpha, G) \).

(b) The existence of a bundle map is shown as in [Die66, p. 5]. (The definition of “locally trivial” in 2.3 is adapted from there.) In order to construct a homotopy between two bundle maps, the proof method from [Die66, p. 6] can be applied to each \( E_j \) separately. The homotopy is compatible with the \((\Gamma, \alpha, G)\)-structure.

(c) See [Dol63, p. 251].

3.2 Henceforth let \( E = E(\Gamma, \alpha, G) \) and \( q : X \to \Gamma/\Lambda \) be a local object. The fibration \( f_q : (X \times E)/G \to \Gamma/\Lambda \) associated to \( q \) with fiber \( E \) is a \( \Gamma \)-map. We say \( f_q \) is \( \Gamma \)-shrinkable (cf. [Dol63, p. 225]) if there exists a \( \Gamma \)-section \( s : \Gamma/\Lambda \to (X \times E)/G \) of \( f_q \) such that \( sf_q \) is \( \Gamma \)-homotopic over \( \Gamma/\Lambda \) (i.e. fiberwise) to the identity of \((X \times E)/G\).

Theorem. For any local object \( q \), \( f_q \) is \( \Gamma \)-shrinkable.

Proof. We describe points in \( E \) via families \((a_{j,n}| j \in J, n = 1, 2, \ldots) \) in which \( a_{j,n} = t_{j,n}x_{j,n}, t_{j,n} \in I, \) and \( x_{j,n} \in X_j \); furthermore only finitely many \( t_{j,n} \) are nonzero and \( \sum t_{j,n} = 1 \). Let \( q : X \to \Gamma/\Lambda \) be isomorphic to \( X_0 \to Y_0, 0 \in J \). We assume that \( X = X_0 \). The space \( X \) is of the form \((\Gamma \ltimes_\alpha G)/H \). Therefore, we describe points from \( X \) by their representatives in \( \Gamma \times G \).

We define a section \( s \) as

\[ s(\gamma\Lambda) = (\gamma, e, a_{j,n}(\gamma)), \]

in which \( a_{0,1}(\gamma) = 1(\gamma, e) \) and \( a_{j,n}(\gamma) = 0(\gamma, e) \) for \((j, n) \neq (0, 1)\), with \( e \in G \) being the identity element. The map \( s \) is well-defined, a \( \Gamma \)-map, and a \( \gamma \)-section.

We construct the desired \( \Gamma \)-homotopy from \( sf \) to the identity as follows. Let \( H_t : (X \times E)/G \to (X \times E)/G \) be defined as

\[ a_{j,n} = \begin{cases} \frac{tt_{j,n}x_{j,n}}{t} & j \neq 0, \\ \frac{tt_{0,n-1}x_{0,n-1}}{t} & j = 0, n > 1, \\ (1 - t)(\gamma, g) & j = 0, n = 1. \end{cases} \]

Then, \( H_0 = df \) and \( H_1 \) is almost equal to the identity, namely in the coordinates with \( j \neq 0 \). Similarly to in [Die66, p. 6], we now construct a homotopy in infinitely many steps which connects \((0x_1, t_1x_2, t_2x_2, \ldots) \in E_0\) to \((t_1x_1, t_2x_2, \ldots) \in E_0\) and provides a homotopy between \( H_1 \) and the identity on \((X \times E)/G\). One checks that these homotopies are compatible with the action of \( \Gamma \) and lie over \( \Gamma/\Lambda \).
3.3 In this manner one can construct universal bundles for restricted categories of \((\Gamma, \alpha, G)\)-bundles. For example: trivial \(\Gamma\)-action on the base space of the bundle; or the base space has finitely many orbits; or the \(\Gamma\)-action on the base space factors through a quotient group of \(\Gamma\). Of course the join in general does not “contain” only the local objects from which it is formed. Therefore the reduced join (in the sense of [Pal60, p. 1.3.6]) is to be used.

In the construction of universal bundles, one also does not have to start from all isomorphism types of local objects. One may restrict to a set of local objects, such that any local object maps to a one in the set.

4 The Homotopy Theorem

Let \(p : X \to B \times I\) be a numerable \((\Gamma, \alpha, G)\)-bundle. Let \(r : B \times I \to B \times I\) be the map \(r(b, t) = (b, 1)\).

**Theorem.** Under the following conditions there exists a \((\Gamma, \alpha, G)\)-bundle map \(p \to p\) lying over \(r\):

(a) All of the orbits of \(B\) consist of finitely many points (e.g., \(\Gamma\) is a finite group).

(b) The space \(B\) is the limit of a sequence \(B_1 \subset B_2 \subset \cdots\) of compact \(\Gamma\)-spaces \(B_i\) and \(G\) is a compact Lie group.

**Consequence.** Let \(p : X \to C\) be a \((\Gamma, \alpha, G)\)-bundle and \(H : B \times I \to C\) be a \(\Gamma\)-homotopy (i.e., \(H_t\) is a \(\Gamma\)-map for all \(t \in I\)). Let the bundle \(H^*(p)\) induced from \(p\) by \(H\) be numerable (e.g., if \(p\) is numerable). Then, under either conditions (a) or (b) of the theorem, the bundle \(H^*(p)\) is equivalent to the product bundle \(H_0^*(p) \times I\). In particular, \(H_0^*(p)\) and \(H_1^*(p)\) are equivalent.

**Proof of the theorem.** (a) In this case the proof in [Hus66, p. 48-50] transfers nearly verbatim. This is essentially because homotopies in the base space of the local objects in question are constant. If we say a bundle is trivial when it admits a map to a local object, one only needs to consider a slightly different proof from [Hus66, Lemma 4.9.4, p. 49].

(b) First, let \(B\) be compact. Let \(r_t : B \times I \to B \times I\) be given by \(r_t(b, s) = (b, t)\), so \(r = r_1\). Let \(g_t : B \times I \to (B \times I) \times (B \times I)\) be given by \(g_t(u) = (u, r_t u)\) and let \(q_t : Y_t \to B \times I\) be the map induced by the product \(p \times p\) through \(g_t\). The map \(q_t\) is a \((\Gamma, \alpha', G \times G)\)-bundle in a canonical way, where \(\alpha' : \Gamma \to \text{Aut}(G \times G)\) is given by \(\alpha'_\gamma(g, h) = (\alpha_\gamma(g), \alpha_\gamma(h))\). Let \(K = \Gamma \times_{\alpha'} (G \times G)\). There is a right \(G \times G\)-action on \(G\) via \(x(g, h) = g^{-1}xh\) and a left \(\Gamma\)-action via \(\gamma x = \alpha_\gamma(x)\). We have

\[\gamma(x(g, h)) = (\gamma x) \cdot \alpha'_\gamma(g, h).\]

In this way \(G\) becomes a \(K\)-space. Bundle maps \(p \to p\) over \(r_t\) correspond to \(\Gamma\)-sections of \(q_t\) and these to \(K\)-maps \(f : Y_t \to G\) (cf. [Hus66, 4.8, p.46-47]). Let \(Y(t)\) be the part of \(Y_t\) lying over \(B \times \{t\}\). We have a \(K\)-map \(\alpha : Y(t) \to G\). It admits an extension from a neighborhood \(Y(t)\) to \(Y_t\) [Pal60, p. 1.6]. From the compactness of \(B\), there is a neighborhood \([a_t, b_t]\) for \(t\) in \(I\), such that a bundle map from \(p\) restricted to \(p^{-1}(B \times [a_t, b_t])\), over \(r_t\) restricted to \(B \times [a_t, b_t]\), exists, which is the identity over \(B \times \{t\}\). Thus, \(p^{-1}(B \times [a_t, b_t])\) is \(K\)-isomorphic over \(B \times [a_t, b_t]\) to \(p^{-1}(B \times \{t\}) \times [a_t, b_t]\). The claim follows easily.

For a sequence \((B_i)\) of compact \(\Gamma\)-spaces, one can construct the bundle map with the previously proven one inductively over \(B_i \times I \cup \lim B_i \times 0\). We do not go any further in detail. \(\square\)
5 Characterization of Universal Bundles

5.1. Theorem. Let \( p : E \to B \) be a \((\Gamma, \alpha, G)\)-bundle, such that for every local object \( q : X \to \Gamma/\Lambda \), the associated bundle \((X \times E)/G \to \Gamma/\Lambda\) is \(\Gamma\)-shrinkable. Then, every numerable \((\Gamma, \alpha, G)\)-bundle admits a bundle map to \( p \) and every two such bundle maps are homotopic.

Proof. Let \( V \to Y \) be a numerable \((\Gamma, \alpha, G)\)-bundle. We consider the associated bundle \((V \times E)/G \to Y\).

Claim: this associated bundle has the \(\Gamma\)-section extension property (\(\Gamma\)-SEP). By [Dol63, p. 2.8], it suffices to show that the local \(\Gamma\)-SEP holds. Locally, the bundle is induced by \(\Gamma\)-shrinkable bundles, which by [Dol63, p. 3.1] have the \(\Gamma\)-SEP. The rest of the proof carries on as in [Dol63, p. 249].

Theorem. Let \( \Gamma \) be finite and \( G \) the limit of a sequence \( G_1 \subset G_2 \subset \cdots \) of Lie groups with finitely many components. A numerable \((\Gamma, \alpha, G)\)-bundle \( E \to B \) is universal, if for every local object \( X \to \Gamma/\Lambda \), the associated bundle \((X \times E)/G \to \Gamma/\Lambda\) is \(\Gamma\)-shrinkable.

Proof. By 2.2, 3.1, and 3.2, there are numerable universal \((\Gamma, \alpha, G)\)-bundles, for which all bundles \((X \times E)/G \to \Gamma/\Lambda\) are \(\Gamma\)-shrinkable. With 4 and 5.1, the proof is continued as in [Dol63, Theorem 7.5, p. 249].

Remark. “Universal” here indicates the validity of the homotopy theorem.

5.3. Example. For \((\mathbb{Z}_2, \alpha, U(n))\)-bundles (cf. 1.3) there are two local objects up to isomorphism. With the notation of 2.1: \( \Lambda = 0 \) and \( t \) trivial; \( \Lambda = \mathbb{Z}_2 \) and \( t \) trivial. For the construction of universal bundles one can reduce to the case of \( U(n) \) over a point with the \(\mathbb{Z}_2\)-action given by conjugation (by the concluding remark in 3.3). The classifying space for these bundles is thus \( BU(n) \) with the involution induced by conjugation.

6 Characteristic Classes

The universal principal \( G \)-bundle \( EG \to BG \) from Milnor [Mil56] admits a \(\Gamma\)-action via

\[
\gamma \cdot (t_1g_1, t_2g_2, \ldots) = (t_1\alpha_\gamma(g_1), t_2\alpha_\gamma(g_2), \ldots).
\]

We denote the \((\Gamma, \alpha, G)\)-bundle given as such by \( p_0 : E_0 \to B_0 \). Let \( j : B_0 \to B(\Gamma, \alpha, G) \) be a classifying map for \( p_0 \). Let \( J = \text{id} \times_{\Gamma} j : E\Gamma \times_{\Gamma} B_0 \to E\Gamma \times_{\Gamma} B(\Gamma, \alpha, G) \). We assume, that the universal \((\Gamma, \alpha, G)\)-bundle is numerable (cf. 3.1).

Theorem. \( J \) and \( j \) (after forgetting the \(\Gamma\)-structure) are homotopy equivalences.

Proof. It follows from 3.2 that \( E(\Gamma, \alpha, G) \) is contractible. Thus, \( p \) is a universal principal \( G \)-bundle by forgetting the \(\Gamma\)-structure. Since \( j \) induces a universal principal \( G \)-bundle from another, it must be that \( j \) is a homotopy equivalence. The claim about \( J \) now follows from [Dol63, Theorem 6.3].
Let $\alpha$ be trivial and be suppressed from the notation. We begin with a numerable $(\Gamma, G)$-bundle $p : V \to X$ over a free $\Gamma$-space $X$ such that the quotient bundle $q : X \to X/\Gamma$ is numerable. Let $k : X \to B(\Gamma, G)$, $k' : X/\Gamma \to BG$, and $(L, l) : (X, X/\Gamma) \to (E\Gamma, B\Gamma)$ be classifying maps from $p, p/\Gamma : V/\Gamma \to X/\Gamma$, and $q$. The associated fibration $r : E\Gamma \times_{\Gamma} X \to X/\Gamma$ is a homotopy equivalence, since the fibers are contractible. A inverse equivalence $s : X/\Gamma \to E\Gamma \times_{\Gamma} X$ is provided by $s(\Gamma x) = (Lx, x) \mod \Gamma$.

**Theorem.** The following diagram commutes up to homotopy:

$$
\begin{align*}
E\Gamma \times_{\Gamma} X & \xrightarrow{id \times rk} E\Gamma \times_{\Gamma} B(\Gamma, G) \\
X/\Gamma & \xrightarrow{[l, k']} B\Gamma \times BG
\end{align*}
$$

**Proof.** First of all, $jk'q$ is $\Gamma$-homotopic to $k$, since both maps induce the same $(\Gamma, G)$-bundle over $X$. Thus, $id \times rjk'q$ is homotopic to $id \times rk$. We assemble $id \times rjk'q$ with $q$ to obtain $[l, k']$. Since $r$ and $s$ are inverse homotopy equivalences, the claim follows.

6.3 If $k$ is a functor defined on the category of topological space, we obtain a functor $r k$ defined on the category of $\Gamma$-spaces from it by defining

$$rk(X) = k(E\Gamma \times_{\Gamma} X).$$

We seek characteristic classes for $(\Gamma, O(n))$-bundles with values in cohomology $rH^*$. The universal classes are thus elements in $rH^*(B(\Gamma, O(n)))$, or—due to the homotopy equivalence from 6.1—elements in $H^*(B\Gamma \times BO(n))$.

Characteristic Classes for bundles over free $\Gamma$-spaces were considered by Conner–Floyd (e.g., in [CF64, p. 91]). The connection between there and the classes considered here is the theorem from 6.2, which gives a splitting of the classes that goes beyond that of 6.1 For instance, using cohomology with coefficients in a field $L$, so that we have the Künneth formula

$$H^*(B\Gamma \times BO(n); L) \cong H^*(B\Gamma; L) \otimes H^*(BO(n); L)$$

at our disposal, all characteristic classes from $V \to X$ are polynomial in characteristic classes from $X \to X/\Gamma$ and $V/\Gamma \to X/\Gamma$.

**Corrections (March 3, 1969).** The homotopy Theorem 4 (and also Theorem 5.2) apply generally for compact Lie groups $\Gamma$. The proof of 4 (a) can be carried out “inductively over $\Gamma,” if one considers the following: Let $p : X \to B \times [0, 1]$ be induced by a local object through $f : B \times [0, 1] \to \Gamma/\Lambda$. Then, $B$ is of the form $\Gamma \times_{\Lambda} Y$ with $Y = f_0^{-1}(\Lambda)$ [Pal60 p. 1.7.9]. The restriction $p'$ of $p$ to $Y \times [0, 1]$ is equivalent to $p_0' \times id$ (this is clear for $\Gamma = \Lambda$ and otherwise via induction [Pal60 p. 1.8.1]). Then, $p = \Gamma \times_{\Lambda} p' = (\Gamma \times p_0') \times id$. The theorem applies also locally. One completes the proof as usual.

**References**


