Chebyshev and Sobolev Orthogonal Polynomials

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Preliminaries

- Let $V_0 = \{q_0, q_1, q_2\} \in \mathbb{R}^2$ and $F_i(x) = \frac{1}{2}(x + q_i)$ for $i = 0, 1, 2$. Then

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- We work on the finite graph approximation $V_m = \bigcup_{|w|=m} F_w(V_0)$

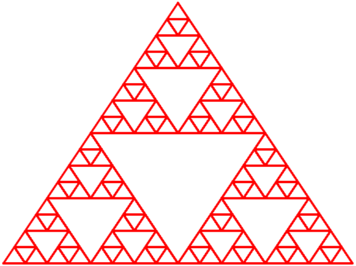

Figure: V_4

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- $\partial_T u(q_i)$ is the **tangential derivative** of u at q_i where

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Polynomials on SG: The space $\mathcal{H}_j$

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- Let $f \in \mathcal{H}_j$. Then f is determined uniquely by the values $(f(q_0), f(q_1), f(q_2), \Delta f(q_0), \Delta f(q_1), \ldots, \Delta^j f(q_2))$

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$$\Delta^m f_{nk}(q_i) = \delta_{mn}\delta_{ki}$$

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- For example, the three harmonic functions h_i where $h_i(q_j) = \delta_{ij}$ form a basis for $\mathcal{H}_0$

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- We want to mimic the case on $I = [0, 1]$ where we can expand a function as a Taylor series in terms of basis functions where we use derivative information at one point.

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$$P_{01} = h_0 + h_1 + h_2 \qquad P_{02} = -\frac{1}{2}(h_1 + h_2) \qquad P_{03} = \frac{1}{2}(h_1 - h_2)$$

If $f \in \mathcal{H}_0$ then $f = f(q_0)P_{01} + \partial_n f(q_0) P_{02} + \partial_T f(q_0) P_{03}$

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  If $f \in \mathcal{H}_0$ then $f = f(q_0)P_{01} + \partial_n f(q_0)P_{02} + \partial_T f(q_0)P_{03}$
- We introduce the following basis $\{P_{jk}\}$ where

$$\begin{aligned} \Delta^n P_{jk}(q_0) &= \delta_{nj}\delta_{k1} \\ \Delta^n \partial_n P_{jk}(q_0) &= \delta_{nj}\delta_{k2} \\ \Delta^n \partial_T P_{jk}(q_0) &= \delta_{nj}\delta_{k3} \end{aligned}$$

This is known as the **monomial basis**.

Polynomials on SG: The monomial basis

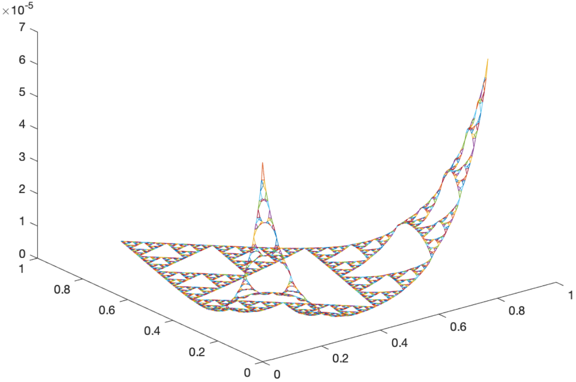

Figure: P_{31}

Polynomials on SG: The monomial basis

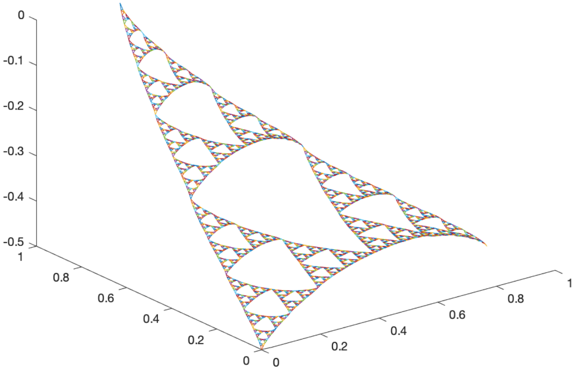

Figure: P_{02}

Polynomials on SG: The monomial basis

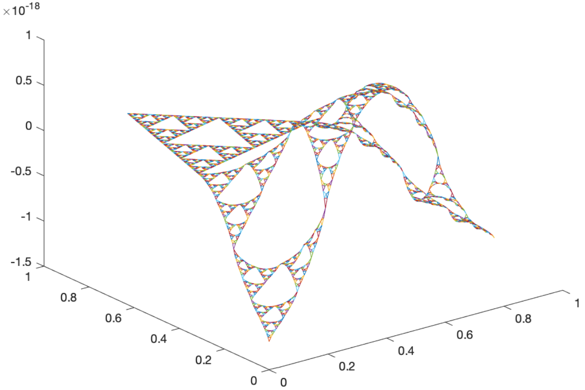

Figure: P_{82}

Polynomials on SG: The monomial basis

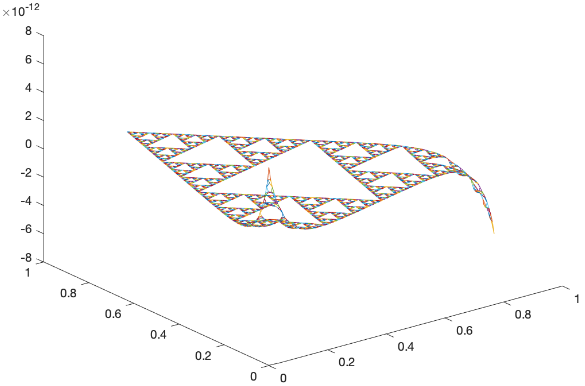

Figure: P_{53}

Polynomials on SG: The monomial basis

- Note that P_{j1} and P_{j2} are symmetric across the line through q_0 and $F_1 q_2$ and P_{j3} is anti-symmetric. We thus refer to them as the *symmetric* and *anti-symmetric* families.

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- Note that P_{j1} and P_{j2} are symmetric across the line through q_0 and $F_1 q_2$ and P_{j3} is anti-symmetric. We thus refer to them as the *symmetric* and *anti-symmetric* families.
- Using the monomial basis, we define can functions as power series expansions:

$$f = \sum_{n=0}^{\infty} \sum_{k=1}^{3} c_{nk} P_{nk}(x)$$

If $|c_{nk}| \leq M r^n$ for some $r < \lambda_2$, the power series converges uniformly. We call such functions **entire analytic**.

Orthogonal polynomials

- The space of polynomials on $\mathbb{R}$, denoted $P(\mathbb{R})$ can be endowed with the following inner product:

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- Common examples for $w(x)$ include $\chi_{[-1,1]}(1-x)^\alpha(1+x)^\beta$, e^{-x^2}, and $\chi_{[0,\infty)} e^{-x}$

Orthogonal Polynomials

$w(x)$	Name
$\chi_{[-1,1]}(1-x)^\alpha(1+x)^\beta$	Jacobi
e^{-x^2}	Hermite
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Table: Classical orthogonal polynomials on $\mathbb{R}$

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Table: Classical orthogonal polynomials on $\mathbb{R}$

There are two special cases of Jacobi polynomials:

- When $\alpha = \beta = 0$, they are known as the **Legendre polynomials**. In this case the inner product is the standard L^2 product on $[-1,1]$: $\langle f, g\rangle_{L^2} = \int_{-1}^{1} fg\,dx$.
- When $\alpha = \beta = 1/2$, they are known as the **Chebyshev polynomials**. Here the weight function is $\sqrt{1-x^2}$

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Okoudjou et. al (2012) found the Legendre polynomials on SG (the orthogonal polynomials with respect to the L^2 inner product). We will study orthogonal polynomials with respect to other inner products, usually involving the Laplacian. These will be known as the **Sobolev** inner products.

Origin of Sobolev Orthogonal Polynomials

- Idea (Lewis): Given $f \in L^2[-1,1]$, find polynomial g with deg $g \leq n$ such that g minimizes the quantity $\|f-g\|_H$, where $\|h\|_H^2 := \int (h^2(x)dx + h'^2(x))dx$

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- Solution: $\widetilde{g} = \sum_{i=0}^{n} \langle f, e_i \rangle_H \, e_i$, $\{e_i\}$ orthonormal basis
- Core idea: Approximate a function by polynomials as close as possible.

Generalization to SG

- Unless specified we will consider the Sobolev inner product $\langle f, g\rangle_H := \langle f, g\rangle_{L^2} + \lambda\langle \Delta f, \Delta g\rangle_{L^2}$ for some nonnegative constant λ, where $\langle f, g\rangle_{L^2} := \int f g \, d\mu$ for a regular Borel probability measure μ that is symmetric with respect to the line passing through q_0 and the midpoint of the side opposing q_0

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- the monic Sobolev polynomials $\{S_{nk}(x; \lambda)\}_{n=0}^{\infty}$(For simplicity, write as $\{S_n\}_{n=0}^{\infty}$), is obtained by Gram-Schmidt Process.

Recurrence (Case 1: k=2 or 3)

Theorem: When $k = 2$ or 3 we have the following recurrence relation for $n \geq -1$, where $S_{-1} := 0$.

$$S_{n+2} - a_n S_{n+1} - b_n S_n = f_{n+2}$$

where $f_{n+2} := \mathcal{G}(p_{n+1})$, and $\mathcal{G}(g)(x) := -\int_{SG} G(x,y) g_{n+1}(y) dy$,

$$a_n = -\frac{\langle f_{n+2} S_{n+1} \rangle_H}{\|S_{n+1}\|_H^2}$$

$$b_n = -\frac{\langle f_{n+2}, S_n \rangle_H}{\|S_n\|_H^2}$$

Recurrence: (Case 1: k=2 or 3)

Remark: When $k = 2$ or 3, the same recursive relation is still valid if we replace the Sobolev inner product by $\langle f, g\rangle_H :=$

$$\langle f, g\rangle_{L^2} + \lambda_1 \langle \Delta f, \Delta g\rangle_{L^2} + \lambda_2\, \varepsilon(f, g) +$$

$$[f(q_0)\, f(q_1)\, f(q_2)] M [g(q_0)\, g(q_1)\, g(q_2)]^T$$

for nonnegative constants λ_1 and λ_2, positive semidefinite 3×3 matrix M

Recurrence (Case 1: k=2 or 3)

Corollary: When $k = 2$ or 3, (a_n, b_n) is the unique solution to the system $a_n S_{n+1}(q_1) + b_n S_n(q_1) = S_{n+2}(q_1)$ and $a_n \partial_n S_{n+1}(q_1) + b_n \partial_n S_n(q_1) = \partial_n S_{n+2}(q_1)$. In particular, the matrix $\begin{bmatrix} S_{n+1}(q_1) & S_n(q_1) \\ \partial_n S_{n+1}(q_1) & \partial_n S_n(q_1) \end{bmatrix}$ is non-singular for any integer $n \geq 0$.

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- **Estimates**: $\|S_n\|_H^2 = \Theta(\lambda)$, $|a_n| = O(\lambda^{-1})$, $|b_n| = \Theta(\lambda^{-1})$

$$\|\Delta S_n\|_{L^2}^2 \leq \lambda^{-1}\|G\|_{L^2}^2\|p_{n-1}\|_{L^2}^2 + \|p_{n-1}\|_{L^2}^2$$

$$\|S_n\|_{L^\infty} \leq C(1+\lambda^{-\frac{1}{2}})\|p_{n-1}\|_{L^2}$$

(C is independent of n and λ)

Recurrence (Case 1: k=2 or 3, Asymptotic)

Theorem: Suppose $k = 2$ or 3. Then for any $n \geq 3$, $S_n(x; \lambda)$ converges to f_n uniformly in x as $\lambda \to \infty$. Consequently $\Delta S_n \to p_{n-1}$ uniformly as $\lambda \to \infty$. Also,

$$\lambda(S_n(\lambda) - f_n) \to -\frac{\langle f_n, f_{n-1}\rangle_{L^2}}{\|p_{n-2}\|_{L^2}^2} f_{n-1} - \frac{\|p_{n-1}\|_{L^2}^2}{\|p_{n-3}\|_{L^2}^2} f_{n-2}$$

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- Requires a conjecture: $\partial_n f_t(q_0) \neq 0$, where $f_t := \mathcal{G}(p_{t-1})$.
- **Theorem**: Let $S_{-1} := 0$, k=1, $f_{n+2} = \mathcal{G}(p_{n+1})$ and suppose that $\partial_n f_{n+2}(q_0) \neq 0$, then
  $S_{n+3} - a_n S_{n+2} - b_n S_{n+1} - c_n S_n = f_{n+3} + d_n f_{n+2}$, The matrix

$$\begin{bmatrix} S_{n+2}(q_1) & S_{n+1}(q_1) & S_n(q_1) \\ \partial_n S_{n+2}(q_1) & \partial_n S_{n+1}(q_1) & \partial_n S_n(q_1) \\ S_{n+2}(q_0) & S_{n+1}(q_0) & S_n(q_0) \end{bmatrix} \tag{1}$$

  is non-singular.

Recurrence (Case 2: k=1, Asymptotics)

Theorem: Assume the normal derivative conjecture is true. Then there exists a sequence of monic polynomials $\{g_n\}_{n=0}^{\infty}$ independent of λ such that for any $n \geq 0$, $deg\, g_n = n$, S_n converges uniformly in x to g_n. And $g_{n+3} + d_n g_{n+2} = f_{n+3} + d_n f_{n+2}$ for any $n \geq 1$. For the basic cases, $g_0 = p_0$, $g_1 = p_1$, $g_2 + d_{-1} g_1 = f_2 + d_{-1} f_1 - \frac{\langle f_2 + d_{-1} f_1, g_0 \rangle_{L^2}}{\|g_0\|_{L^2}^2} g_0$, and $g_3 + d_0 g_2 = f_3 + d_0 f_2 - \frac{\langle f_3 + d_0 f_2, g_0 \rangle_{L^2}}{\|g_0\|_{L^2}^2} g_0$. Moreover, for any $\alpha < 1$, $n \geq 0$, $\lim_{\lambda \to \infty} \lambda^{\alpha} (S_n(\lambda) - g_n) = 0$ uniformly in x.

Recurrence: Generalization (k=2 or 3)

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- **Theorem**: $S_{n+m+1} - \mathcal{F}_{n+m+1} - \sum_{l=0}^{2m-1} a_{n,l} S_{n+m-l} = 0$, where $\mathcal{F}_{n+m+1} := \mathcal{G}^m p_{n+1}$

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- **Remark**: It is still true if we consider $\langle f, g\rangle_{H^m} = \sum_{l=0}^{m} \lambda_l \int_{SG} \Delta^l f \Delta^l g \, d\mu + \sum_{l=0}^{m-1} \beta_l \, \varepsilon(\Delta^l f, \Delta^l g) + \sum_{l=0}^{m-1} [\Delta^l f(q_0) \; \Delta^l f(q_1) \; \Delta^l f(q_2)] M_l [\Delta^l g(q_0) \; \Delta^l g(q_1) \; \Delta^l g(q_2)]^T$, where M_l are positive definite 3×3 matrices.

Recurrence: Generalization (k=2 or 3)

Asymptotic: we consider the case $\lambda_m \to \infty$, and the other parameters are bounded.

Theorem: Suppose $k = 2$ or 3, and there exists $M > 0$ such that $\lambda_l \leq M$ for any $l < m$. Then for any $n \geq 2m + 1$, we have

$$\|S_n - \mathcal{F}_n\|_{L^2} \leq C(n, M, m, \mu)\lambda_m^{-1}$$

Consequently, $\lim\limits_{\lambda_m \to \infty} \|\Delta^i S_n - \mathcal{G}^{m-i} p_{n-m}\|_{L^\infty} \to 0$ for any $0 \leq i \leq m$.

Zeros: Continuous functions

Theorem (Topological result): Let f be a continuous function defined on SG. Suppose f has finitely many zeros. Let Z_0 be the intersection of zero set Z of f and V^*. Then for any connected component D in $SG \setminus Z_0$, either $f \geq 0$ on D or $f \leq 0$ on D.

Zeros: Entire Analytic functions

Theorem: Suppose

$f(x) = \sum_{j=t_1}^{\infty} c_{j1} P_{j1}^{(0)}(x) + \sum_{j=t_2}^{\infty} c_{j2} P_{j2}^{(0)}(x) + \sum_{j=t_3}^{\infty} c_{j3} P_{j3}^{(0)}(x)$ where $c_{t_1,1}$, $c_{t_2,2}$ and $c_{t_3,3}$ are nonzero and has zero set Z. Then

1. Z is compact and nowhere dense in SG.

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1. Z is compact and nowhere dense in SG.
2. If $t_3 < t_1 - 1$ and $t_3 < t_2$, then f has infinitely many zeros that has limit point q_0. Moreover, suppose the conjecture $P_{j1} > 0$ is true, $t_1 \leq t_2$ and $t_1 \leq t_3$. Then q_0 has a neighborhood U such that $Z \cap U \subset \{q_0\}$.

Zeros of OPs

How many zeros do the Orthogonal Polynomials have?

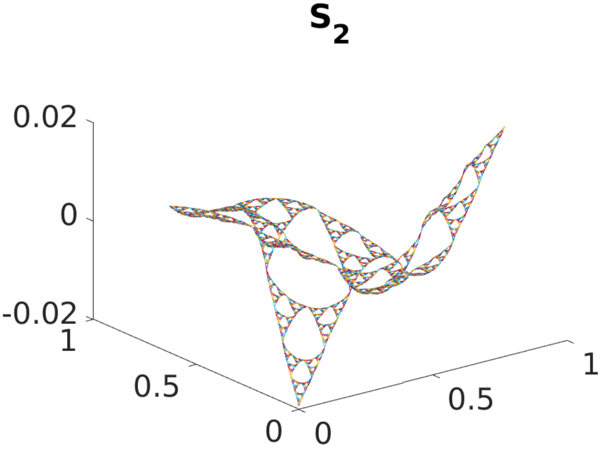

Figure: Antisymmetric Sobolev OP of Degree 2

Counting Zeros

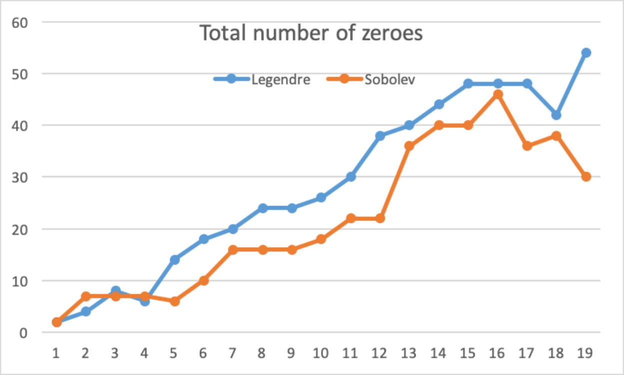

Figure: Edge Zeros of Orthogonal Polynomials

Notice that some polynomials have more than $3n + 3$ zeros...

Polynomial Interpolation and Quadrature on SG

Questions:

- Can we use polynomials (orthogonal or otherwise) to accurately interpolate functions?
- Can we obtain an analog of Gauss-Legendre quadrature on SG?
- Can we develop an algorithm for polynomial quadrature on SG and determine error estimates for it?

Interpolation

- Polynomials of degree n can have more than $3n+3$ zeros on SG.
- We need to establish the invertibility of

$$M_n = \begin{bmatrix} P_1(x_1) & P_1(x_2) & \dots & P_1(x_{3n+3}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{3n+3}(x_1) & P_{3n+3}(x_2) & \dots & P_{3n+3}(x_{3n+3}) \end{bmatrix}$$

Interpolation

If we choose a rotationally symmetric set of points, the interpolation matrix M_n becomes circulant-block. This enables us to prove the invertibility of M_2.

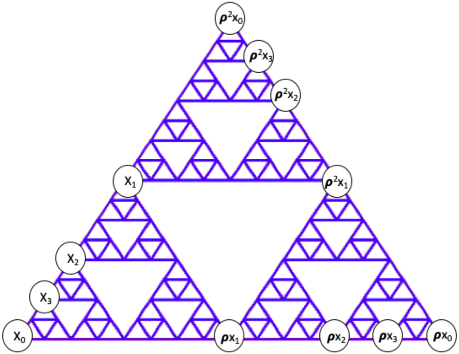

Figure: Interpolation Set for $n = 3$

Interpolation

n	$\lvert M_n \rvert$
1	-1.744e-06
2	1.066e-19
3	-4.200e-41
4	-2.058e-69
5	6.788e-110
6	-1.347e-163
7	5.044e-232
8	-4.976e-316

Table: Determinants of M_n

Interpolation

- The interpolation matrix for the P_{jk} polynomials on V_1 points is invertible (by computation)
- By choosing $n+1$ points on the left side of V_n along with their rotations, the interpolation matrix becomes circulant-block. These matrices are computationally invertible up to at least $n = 50$.
- The general case is unknown.

Quadrature

- Gauss-Legendre quadrature on $\mathbb{R}$ requires the polynomial division algorithm on $\mathbb{R}$. However, we do not have this on SG.
- We have a pseudo-division algorthim on SG, but the quotient is a linear combination of powers of the Green's operator:

$$\mathcal{Q}_f = \frac{1}{b_m} \sum_{i=0}^{n-m} c^{(i)} \mathcal{G}^{(n-m-i)}$$

Quadrature

- We can try to use n-Harmonic spline quadrature, but this comes back to the interpolation problem since we need to solve

$$\begin{bmatrix} P_1(x_1) & P_1(x_2) & \dots & P_1(x_{3n+3}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{3n+3}(x_1) & P_{3n+3}(x_2) & \dots & P_{3n+3}(x_{3n+3}) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_{3n+3} \end{bmatrix} = \begin{bmatrix} \int_{SG} P_{01} \\ \vdots \\ \int_{SG} P_{n3} \end{bmatrix}$$

N-Harmonic Extension: The Solution to Many Polynomial Problems on SG

- Is there an algorithm to extend a function defined on $3n+3$ vertices of V_n n-Harmonically?
- Given this algorithm, we have the following quadrature error estimate:

$$\left| I_n^m(f) - \int_{SG} f \right| \leq c_1 5^{-(n+1)(m-n)} \|\Delta^{(n+1)} f\|_\infty$$

Outstanding Questions Regarding Polynomials on SG

1. Is $\partial_n f_n \neq 0 \ \forall \ n$?
2. Is $P_{j1} > 0$ except at q_0?
3. Interpolation problem: Does sampling an n degree polynomial on any $3n+3$ points uniquely determine the polynomial?

Questions 2 and 3 can be solved given an n-Harmonic Extension Algorithm.

Extremal Points of Polynomials

Extremal Points of Polynomials

Definition of local extrema:

1. For a function u defined on SG and $x \in SG$, we say that x is a **local maximum (minimum)** of u if $\exists$ neighborhood U s.t. $x \in U \subseteq SG$ and $\forall y \in U$, we have $u(x) \geq u(y)$ (or $u(x) \leq u(y)$).

2. For a function u defined on SG and $x = F_w q_n = F_{w'} q_{n'} \in SG$, we say that x is a **local edge maximum (minimum)** of u if $\exists$ M s.t. $\forall m \geq M$, $u(x) \geq u(F_w F_n^m q_j)$ for $j = n-1, n+1$ and $u(x) \geq u(F_{w'} F_{n'}^m q_{j'})$ for $j' = n'-1, n'+1$. [u is larger than a discrete set of points on all neighboring edges]

$\Rightarrow$ Local edge extrema are weaker than local extrema

Extremal Points of Polynomials - General Case

Theorem:
(Necessary conditions for x to be a local edge extrema of u)

1. If $x \in V_0$ is on the boundary, then $\partial_n u(x) \geq 0$ if x is a local maximum (or $\partial_n u(x) \leq 0$ for x a local minimum)

2. If x is not on the boundary, then $\partial_n u(x) = 0$, and $\Delta u(x) \leq 0$ if x is a local maximum (or $\Delta u(x) \geq 0$ if x is a local maximum)

Corollary: $P_{11} \geq 0$

Extremal Points of Polynomials - General Case

Theorem:
(Sufficient conditions for x to be a local edge extrema of u)

1. Let $u \in dom\Delta_\mu$ and $x = F_w q_n = F_{w'} q_{n'}$ be a junction point

   satisfying $\begin{cases} \Delta u(x) < 0 \text{ (or } \Delta u(x) > 0) \\ \partial_n u(F_w q_n) = \partial_n u(F_{w'} q_{n'}) = 0 \\ \partial_T u(F_w q_n) = \partial_T u(F_{w'} q_{n'}) = 0 \end{cases}$

   Then t is a local edge maximum (or minimum) of u on SG.

$\Rightarrow$ This only holds for local edge maximum, not local maximum.
(Normal derivatives and tangential derivatives are very weak characterizations of local behavior)

Extremal Points of Polynomials - Harmonic Case

Lemma: (Behavior of Harmonic Function on Outmost Edges)
Let h be a harmonic function on SG, and we consider the edge between $q_0, q_1 \in V_0$, assuming $h(q_0) \leq h(q_1)$.

1. If $\partial_n h(q_0) \cdot \partial_n h(q_1) \leq 0$, then h is increasing from q_0 to q_1.
2. If $\partial_n h(q_0), \partial_n h(q_1) > 0$, then h first decrease then increase from q_0 to q_1.
3. If $\partial_n h(q_0), \partial_n h(q_1) < 0$, then h first increase then decrease from q_0 to q_1.

$\Rightarrow$ Behavior of harmonic functions on edges is completely characterized by sign of normal derivatives on V_0.

Extremal Points of Polynomials - Harmonic Case

Theorem: (Local Extrema of Harmonic Functions)
Let h be a non-constant harmonic function: $h(q_0) = \alpha$, $h(q_1) = \beta$, $h(q_2) = \gamma$ with $\alpha \leq \beta \leq \gamma$ not all equal.

1. If $\partial_n h(q_1) = 0$, then q_0 is the unique local minimum and q_2 is the unique local maximum.
2. If $\partial_n h(q_1) < 0$, then q_0, q_1 are the only local minima and q_2 is the unique local maximum.
3. If $\partial_n h(q_1) > 0$, then q_0 is the unique local minimum and q_1, q_2 are the only local maxima.

Extremal Points of Polynomials - Biharmonic Case

Theorem:
(Necessary conditions for local extrema of biharmonic functions)

Let $u \in \mathcal{H}^1$ be a nonconstant biharmonic function on SG, and $x = F_w q_n = F_{w'} q_{n'}$ be a junction point that is a local extrema of u. Then we have:

1. $\partial_n u(x) = 0$.
2. Either $\Delta u(x) \neq 0$ or $\partial_n \Delta u(x) \neq 0$.

$\Rightarrow$ Proof: From the properties of antisymmetric functions.

Extremal Points of Polynomials - Biharmonic Case

Theorem:
(Sufficient conditions for local extrema of biharmonic functions)

Let $u \in \mathcal{H}^1$ be a function on SG, and $x = F_w q_n = F_{w'} q_{n'}$ be a junction point. Suppose $\begin{cases} \partial_n u(x) = 0 \\ \partial_T u(x) = 0 \\ \partial_n \Delta u(x) = 0 \\ \partial_T \Delta u(x) = 0 \end{cases}$, then x is a local optimum of u.

$\Rightarrow$ This comes from the property that P_{11} achieves global maximum/minimum on the boundary.

Extremal Points of Polynomials - Summary and Questions

Recap:

1. Define local extrema and local edge extrema
2. Local edge extrema + functions in the domain of Laplacian
3. Local extrema + harmonic/biharmonic functions

Questions to consider:

1. Can any of the above be generalized to n-harmonic functions?
2. Is it possible to design an efficient algorithm to find local extrema of n-harmonic functions, given that we can evaluate the n-jet at all points?

Chebyshev Polynomials on SG

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Definition of Chebyshev Polynomials on $[-1,1]$:
The n^{th} **Chebyshev polynomial** $T_n(x) : [-1,1] \to \mathbb{R}$ is defined as
$T_n(x) := \cos(n \cos^{-1}(x))$

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An important property of Chebyshev Polynomials on $[-1, 1]$ is the extremal principle:
$\forall P(x) : [-1, 1] \to \mathbb{R}$, monic polynomial of degree n,
$||2^{1-n}T_n(x)||_u \leq ||P(x)||_u$, where $||\cdot||_u$ is the uniform norm of functions.

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Remark: the leading coefficient of $T_n(x)$ is 2^{1-n}, and hence $2^{1-n}T_n(x)$ is the monic polynomial on $[-1, 1]$ that minimizes the uniform norm.

Chebyshev Polynomials on SG

Chebyshev Polynomials on SG

The revised definition of Chebyshev Polynomials on any compact $K \subseteq \mathbb{R}$:
The n^{th} **Chebyshev polynomial** $T_n(x) : K \to \mathbb{R}$ is defined as the monic polynomial of degree n that has the smallest uniform norm of all monic polynomial of degree n.

Chebyshev Polynomials on SG

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Fix k, then **the monic polynomial of degree** j is a polynomial of the form $\sum_{l=0}^{j} c_l P_{lk}$, where $c_j = 1$.

Chebyshev Polynomials on SG

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Fix k, then **the monic polynomial of degree** j is a polynomial of the form $\sum_{l=0}^{j} c_l P_{lk}$, where $c_j = 1$.

Definition of the j^{th} Chebyshev Polynomials of family k on SG:
Fix $k = 1, 2, 3$, then **the j^{th} Chebyshev Polynomials of family k,** $T_{jk}(x)$, is the monic polynomial of degree j, such that $\forall P(x)$ a monic polynomial of degree j, $||T_{jk}||_u \leq ||P||_u$

Partial Results on $\mathcal{H}_1$

Partial Results on $\mathcal{H}_1$

The problem right now reduces to for fixed k, find a_k, such that $P_{1k}(x) + a_k P_{0k}(x)$ has the smallest uniform norm of all monic polynomial of degree 1.

- For the 1-family, we have an exact answer, that $T_{11}(x) = P_{11}(x) - \frac{1}{12} P_{01}(x)$
- This is because $P_{11} \geq 0$ and hence it achieves the minimum value 0 at q_0 and the maximum value $\frac{1}{6}$ at q_1 and q_2.
- Unfortunately, the proof that $P_{11} \geq 0$ is overcomplicated and cannot be generalized to arbitrary j.

Image of $T_{11}(x)$

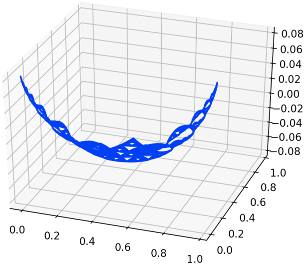

Figure: Plot of Chebyshev polynomial of order 1 of family 1, with $a = -\frac{1}{12}$. The boundary node on the left is q_1, the one on the right is q_2, and the boundary node on the back is q_0.

Partial Results on $\mathcal{H}_1$

- For the 2^{nd} Chebyshev polynomials of family 2 and the family 3, we only have experimental result, and our experiments show that $a_2 = 0.0619339$ and $a_3 = 0.0275013$
- We found those values by firstly determining loose bounds of a_2 and a_3, which are $[-\frac{2\beta_1}{\beta_0}, 0] = [-\frac{8}{45}, 0]$ for a_2 and $[-\frac{2\alpha_1}{\alpha_0}, 0] = [-\frac{1}{15}, 0]$ for a_3.
- Then we partition the intervals, test out each a_k, and look for the a_k that gives the smallest uniform norm.

Image of $T_{12}(x)$

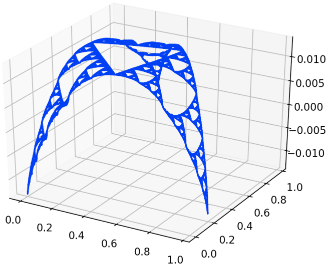

Figure: Plot of Chebyshev polynomial of order 1 of family 2, with $a \approx -0.0619339$. The boundary node on the left is q_1, the one on the right is q_2, and the hidden boundary node on the back is q_0.

Image of $T_{13}(x)$

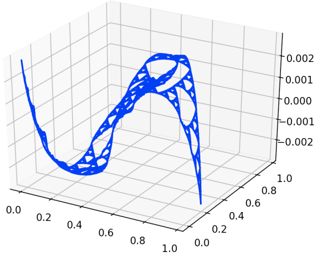

Figure: Plot of Chebyshev polynomial of order 1 of family 3, with $a \approx -0.02750235$. The boundary node on the left is q_1, the one on the right is q_2, and the hidden boundary node on the back is q_0.

Alternating Property of Chebyshev Polynomials

- A degree n polynomial $P_n(x)$ defined on a compact set $K \subseteq \mathbb{R}$, has **an alternating set**, if $\exists \{x_j\}_{j=0}^{n}$ with $x_0 < x_1 < ... < x_n$, so that $P_n(x_j) = (-1)^{n-j}||P_n(x)||_u$.
- **The Alternation Theorem**: A monic polynomial of degree n is the Chebyshev polynomial if and only if it has an alternating set.
- The experimental results also show that the absolute value of the minimum and the maximum of the monic polynomials become closer when a_2 and a_3 approach the values that minimize their uniform norms.

Alternating Property of Chebyshev Polynomials

- Assume that there exist an a, such that $Q(x) := P_{13}(x) + aP_{03}(x)$ achieves maximum norm at two distinct points $y \in \bigcup_{m=0}^{\infty} F_0^m F_1 SG$ and $z \in \bigcup_{m=0}^{\infty} F_0^m F_1 SG$, and $z = -y$. Then $Q(x)$ is the 1^{st} Chebyshev polynomial of the 3-family.
- Assume $Q(x)$ is not the first Chebyshev polynomial of the 3-family. Then $||T_{13}||_\infty < ||Q||_\infty$. This implies that $|T_{13}(x)| < |Q(x)|$ at y and z. Thus $T_{13} - Q(x)$ cannot be both positive or negative at y and z. Since both $T_{13}(x)$ and $Q(x)$ are monic, $T_{13} - Q(x)$ is spanned by P_{03}, and hence $T_{13} - Q(x)$ has to be both positive or negative at y and z. We have a contradiction.

Further Questions

- Find explicit formulas for Chebyshev polynomials of any degree.
- Replicate the alternation theorem to polynomials on SG.
- Study the orthogonality.
- Find the recurrence relation, if any.

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Questions?

Scalarproduct in $L^2[a, b]$ exists

$Functionalanalysis:$