Consider the diffeomorphism group, as well as the mapping class group of an orientable Riemann surface S possibly with boundary components as follows. If $S_{g,1}$ denotes a surface of genus g with 1 boundary component, consider the group of orientation preserving diffeomorphisms $Diff(S_{g,1})$ for which diffeomorphisms are required to be the identity on the boundary. The mapping class group $\pi_0 Diff(S_{g,1})$, denoted $\Gamma_{g,1}$, is the group of isotopy classes of diffeomorphisms fixing the boundary pointwise.

Gluing in the natural way gives stabilization maps

$$Diff(S_{g,1}) \to \ Diff(S_{g+1,1}),$$

and

$$\Gamma_{g,1} \to \ \Gamma_{g+1,1}.$$

Passage to colimits gives the stable mapping class group

$$\Gamma_\infty = colim \Gamma_{g,1}.$$

A result due to I. Madsen and M. Weiss is an identification of the homotopy type of $B\Gamma_\infty^+$, the plus construction applied to the classifying space. Their result is stated as follows: There is a map

$$\Theta : B\Gamma_\infty \to \ \Omega^\infty \mathbb{CP}^\infty_{-1}$$

which, after passage to $B\Gamma_\infty^+$, is a homotopy equivalence. Their proof, geometric in nature, is to show that the map Θ induces an isomorphism on oriented bordism.

In addition, the construction of the map Θ arises from the natural geometry of $BDiff(S_{g,1})$ classifying smooth surface bundles with fibre $S_{g,1}$. The space $\Omega^\infty \mathbb{CP}^\infty_{-1}$ is also obtained in a natural way. One description in terms of other spaces is that there is a natural map

$$\delta : Q(\Sigma(\mathbb{CP}^\infty_{-1})) \to \ QS^0$$

with homotopy theoretic fibre $\Omega^\infty \Sigma \mathbb{CP}^\infty_{-1}$. Thus $B\Gamma_\infty^+$ is the homotopy theoretic fibre of $\Omega(\delta)$.

One consequence is a determination of the homology of the stable mapping class group.

Much of their work depends directly on $BDiff(S_{g,1})$, and the identification with spaces of surface bundles. The identification with mapping class groups, as well as the utility arises in the following two distinct ways in their work:

1. A theorem due to Earle, and Eells gives that each component of $Diff(S_g)$ is contractible if $g \geq 2$, thus the natural map $BDiff(S_g) \to \ B\Gamma_{g,1}$ is a homotopy equivalence.
2. Harer's stability theorem gives an isomorphism in singular homology $B\Gamma_{g,1} \to B\Gamma_g$, and $B\Gamma_{g,1} \to \ B\Gamma_{g+1,1}$ through the "stable range".

References

[1] I. Madsen, and M. Weiss, to appear.