Questions regarding the stable homotopy of $\mathrm{AutF_n}$

Igusa in his recent book [AMS 2002] proves that the linearization map

$$L\colon \mathrm{AutF_n} \longrightarrow \mathrm{GL_n}\mathbf{Z}$$

induces the trivial map on rational homology in degrees k with $n \geq 2k+1$. Here $\mathrm{AutF_n}$ denotes the automorphism group of a free group with n generators, and $\mathrm{GL}_n\mathbf{Z}$ is the general linear group of rank n with integer coefficients.

He proves actually a much stronger, deep result:

Theorem [Igusa 2002] The induced map of infinite loop spaces

$$\mathbf{Z} \times \mathbf{B}\mathrm{Aut}_\infty^+ \longrightarrow \mathbf{Z} \times \mathbf{B}\mathrm{GL}_\infty\mathbf{Z}^+ =: \mathbf{K}(\mathbf{Z})$$

factors through the space $\Omega^\infty S^\infty \simeq \mathbf{Z} \times \mathbf{B}\boldsymbol{\Sigma}_\infty^+$.

Here $\mathrm{Aut}_\infty = \lim_\mathrm{n} \mathrm{AutF_n}$, $\mathrm{GL}_\infty\mathbf{Z} = \lim_\mathbf{n} \mathrm{GL}_\mathbf{n}\mathbf{Z}$, and $\Sigma_\infty = \lim_n \Sigma_n$ is the infinite symmetric group. The space $K(\mathbf{Z})$ represents the algebraic K-theory of the integers, and X^+ denotes in all cases Quillen's plus-construction with respect to the maximal perfect subgroup which has the same homological properties as X. The above statement (1) now follows immediately from Hatcher's homology stability theorem [Comm. Math. Helv. 1995]. There he also notes that as an immediate consequence of Waldhausen's theory the map of infinite loop spaces induced by the natural inclusion $\Sigma_n \to \mathrm{AutF_n}$ has a splitting; i.e. for some infinite loop space W,

$$\mathbf{Z} \times \mathbf{B}\mathrm{Aut}_\infty^+ \simeq \Omega^\infty \mathrm{S}^\infty \times \mathrm{W}.$$

Igusa's theorem and Hatcher's result (2) contain all the information that I know of on the stable homology of $\mathrm{AutF_n}$. Both rely on the homotopy machinery developed for the study of high dimensional manifolds and use techniques and results from Waldhausen K-theory. In the case of the mapping class group the stable homotopy theoretic approach by work of myself, Madsen and Weiss has by now completely determined the stable homotopy of mapping class groups as was highlighted in Fred Cohen's contribution to this panel discussion.

Question 1:

The linearization map L factors through Waldhausen's

$$A(*) := \mathbf{Z} \times (\mathbf{B}\widehat{\mathrm{GL}}_\infty \boldsymbol{\Omega}^\infty \mathbf{S}^\infty)^+ \simeq \mathbf{Z} \times \mathbf{B} \lim_\mathbf{k} \mathrm{HE}(\vee_\infty \mathrm{S^k})^+.$$

We can think of $A(*)$ as the algebraic K-theory of the ring up to homotopy $\Omega^\infty S^\infty$. The degree of a self-map of a sphere defines a ring map to $\mathbf{Z}$ inducing a natural map on K-theory $A(*) \to K(\mathbf{Z})$. The second equivalent definition of $A(*)$ gives us a natural map

to it from $B\mathrm{Aut}_\infty = \mathrm{BHE}(\vee_\infty \mathrm{S}^1)$; here HE(X) denotes the space of homotopy equivalences of X.
Does the map $\sigma : \mathbf{Z} \times \mathbf{B}\mathrm{Aut}_\infty^+ \to \mathrm{A}(*)$ *factor through* $\Omega^\infty S^\infty$*?*

Discussion:

The map $A(*) \to K(\mathbf{Z})$ is well-known to be a rational equivalence, and hence the map σ is rationally trivial by Igusa's theorem. Furthermore, the analogue for the mapping class group holds; in other words, σ factors when restricted to the image of $\mathbf{Z} \times \mathbf{B\Gamma}_\infty^+ \to \mathbf{Z} \times \mathbf{B}\mathrm{AutF}_\infty^+$ induced by the canonical map $\Gamma_{g,1} \to \mathrm{AutF}_{2\mathrm{g}}$. My proof of this [Proc. Phil. Soc. Cam. 1999] based on work by Dwyer, Williams and Weiss is in spirit closely related to Igusa's book. The same argument would give a positive answer to the question if AutF_n can be realized as a subgroup (up to coherent homotopies) of the diffeomorphism group of some manifold.

Question 2:
Does the map $\phi : \mathbf{Z} \times \mathbf{B\Gamma}_\infty^+ \to \mathbf{Z} \times \mathbf{B}\mathrm{Aut}_\infty^+$ *factor through* $\Omega^\infty S^\infty$*?*

Discussion: One might expect this map to be at least rationally trivial. By the solution of

the Mumford conjecture by Madsen and Weiss the map

$$\mathbf{Z} \times \mathbf{B\Gamma}_\infty^+ \longrightarrow \mathbf{\Omega^\infty S^\infty} \times \mathbf{\Omega^\infty S^\infty}(\mathbf{CP}^\infty)$$

is a rational equivalence where the map to the second factor is induced by the classifying map of the tangent bundle of the underlying surface. (For a description of this map see my paper with Madsen [Inventiones 2001].) Note that the second factor contains all the rational information. But any tangent information is of course lost when passing from diffeomorphisms to homotopy equivalences.

Question 3: *Is the space* W *in (2) rationally trivial? or even contractible?*

Discussion: Even if both questions above can be answered positively this might still be unreasonable to expect, or not?

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