WORKSHOP
MAPPING CLASS GROUPS
VS.
KLEINIAN GROUPS:

analogies and relations.

(a very biased survey).

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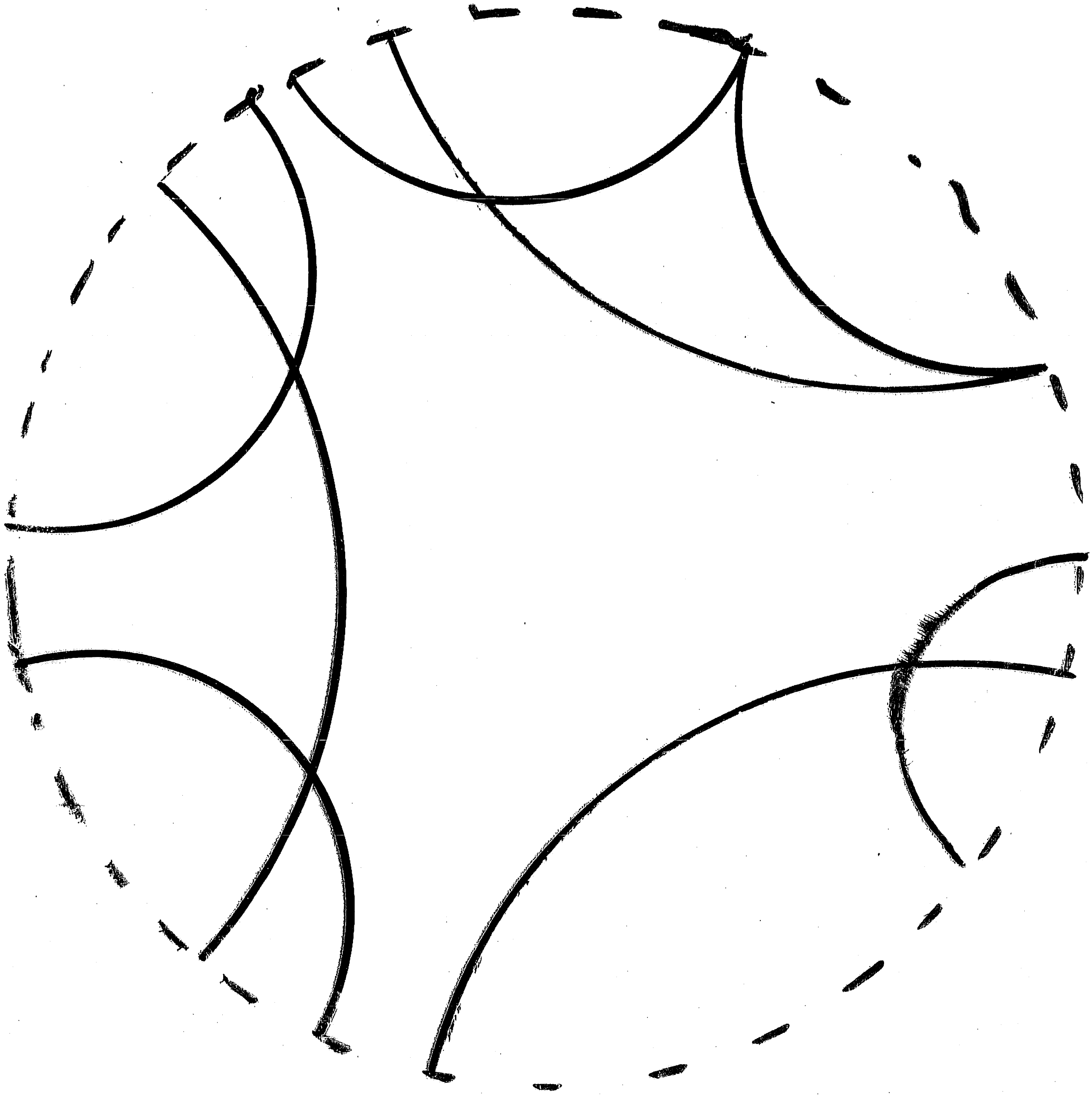
$\mathbb{H}^n$ = hyperbolic space



Complete, simply connected, Riemannian manifold w/ sectional curvature $\equiv -1$

$$\mathbb{H}^n \cong \mathbb{B}^n, \quad ds_H^2 = \frac{4\, ds_E^2}{(1-|x|^2)^2}$$

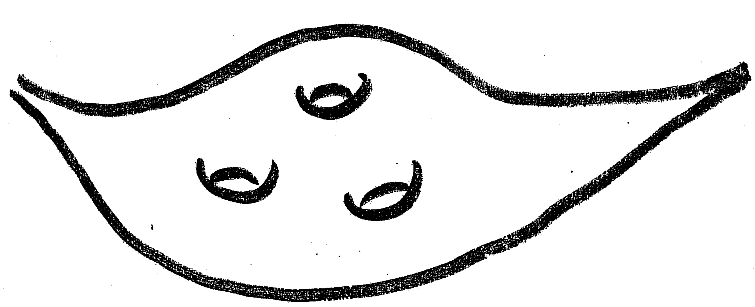
Kleinian Group:

$G < \mathrm{Isom}^+(\mathbb{H}^n)$ discrete

$\Longrightarrow$ $G \curvearrowright \mathbb{H}^n$ properly discontinuously by isometries

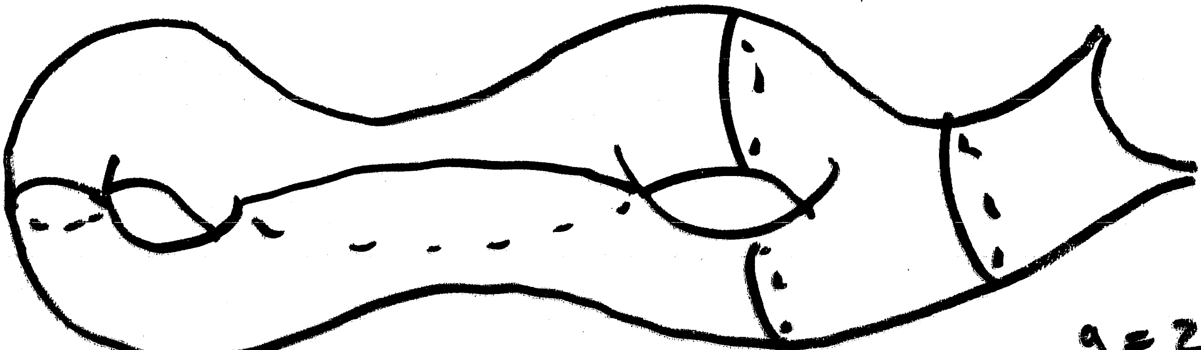
$\mathbb{H}^n/G$ = quotient orbifold

If $\mathbb{H}^n/G$ has finite volume, G is a lattice.



SURFACES

$S = S_{g,n}$



$g=2$
$n=2$

g = genus
n = # punctures

Complexity: $\xi(S) = 3g-3+n$

assume $\xi(S) > 0$.

[Can usually think S closed]

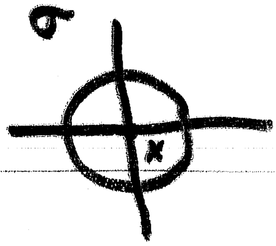
$\mathcal{T}(S) =$ complex/conformal/hyperbolic structures on S up to $\mathrm{Homeo}_0(S)$ $(\simeq \mathrm{id}_S)$

= Teichmüller Space of S.

$d_{\mathcal{T}}$ = Teichmüller metric

measures difference between conformal structures

σ



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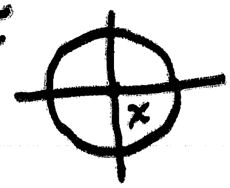
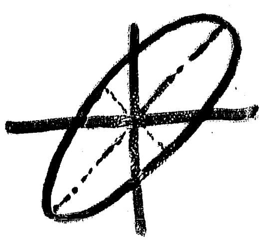




$$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S))$$
$$= \text{Mapping Class group of } S.$$

- (Fricke): $\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$ is properly discontinuous.

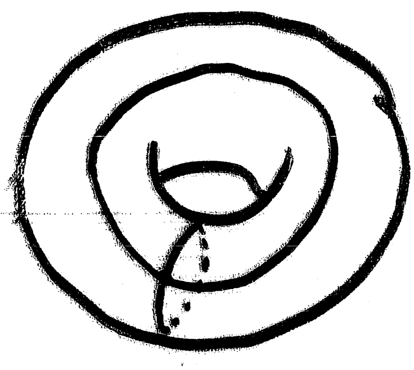
- Acts by isometries.

- (Teichmüller, Ahlfors, Bers): $(\mathcal{T}, d_{\mathcal{T}})$ is a complete, unique geodesic space, $\mathcal{T}(S) \cong B^{2\xi(S)}$

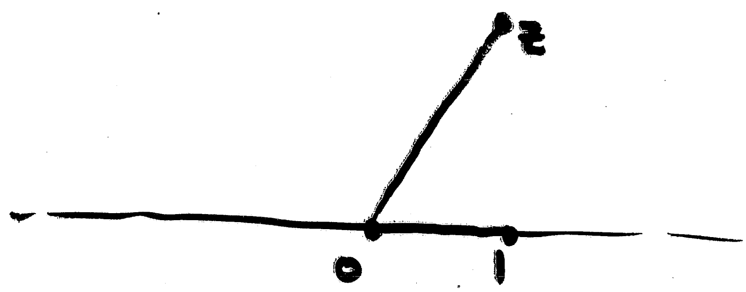
- (Masur): $\mathcal{T}(S)/\mathrm{Mod}(S)$ $(\cong \mathcal{M}_{g,n})$ has finite volume.....

$S_{1,1}$, $S_{0,4}$ $(\& \, S_{1,0})$

$$\begin{array}{ccc} \mathcal{T}(S) & \underset{\text{isom}}{\cong} & \mathbb{H}^2 \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Mod}(S) & \cong & (P)SL_2\mathbb{Z} \end{array}$$



$\mathbb{H}^2 \subset \mathbb{C}$



CAN TRY TO PURSUE THE ANALOGY:

- $G \curvearrowright \mathbb{H}^n$
- lattice in $\mathrm{Isom}^+ \mathbb{H}^n$

$\rightsquigarrow$ $\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$

WHY?

- HOMOMORPHISMS

$$\phi : \pi_1 X \to \mathrm{Mod}(S)$$

describe S-bundles $/X$ ...

What does action on $\mathcal{T}(S)$ tell us?

- Geometric, Topological, Dynamical aspects of surfaces and their homeomorphisms (and related flows, ...) are reflected in $\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$ ...

- LOTS OF PEOPLE CARE ABOUT $\mathcal{M}_{g,n}$

IN GENERAL:

$\mathcal{T}(S)$ is **not** negatively curved in any sense (Masur, Masur-Wolf)

DIRECT ANALOGY IS FREQUENTLY WRONG

CAN OFTEN USE GEOMETRY OF SURFACE TO FILL IN GAPS.

$\underline{\underline{\mathbb{H}^n}}$: Ideal boundary S^{n-1}_∞ (visual)

$\mathbb{H}^n \cup S^{n-1}_\infty \cong \overline{\mathbb{B}}^n$

ISOMETRIES EXTEND $\rightsquigarrow$ $\mathrm{Isom}^+(\mathbb{H}^n) \cong \mathrm{Conf}(S^{n-1}_\infty)$

$\underline{\underline{\mathcal{T}(S)}}$: $\underline{\underline{\text{Thurston's}}}$ boundary $\mathcal{PMF}(S)$

- $\mathcal{T}(S) \cup \mathcal{PMF}(S) \cong \overline{\mathbb{B}}^{2\xi(S)}$ (Thurston)
- $\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S) \cup \mathcal{PMF}(S)$
- NOT visual boundary (Kerckhoff)
- $\underline{\text{IS}}$ a.e. (Masur)

$\underline{\underline{Q}}$: What happens in the "other" directions.

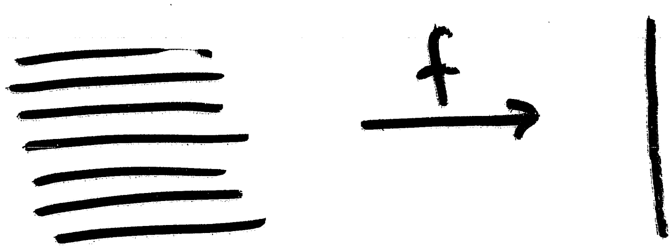
see: Masur, Lenzhen, Rafi for more...
?

MEASURED FOLIATIONS

(Thurston)

Singular foliation $\mathcal{F}$, and a transverse measure of full support.

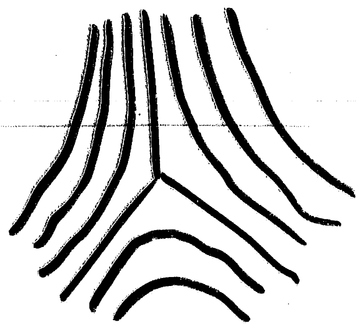
$\underline{\mathcal{F}}$:



decomposition into "leaves", locally level sets of submersion

$\underline{\mu}$: submersions well-defined up to composing with an isometry $h : \mathbb{R} \to \mathbb{R}$

Near singularity or puncture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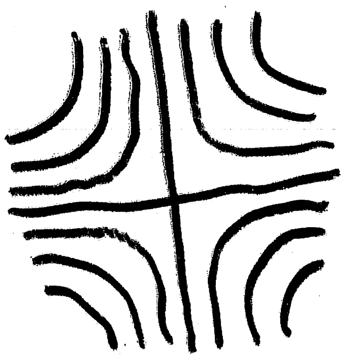
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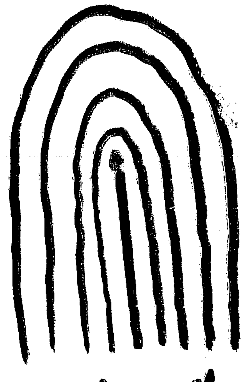
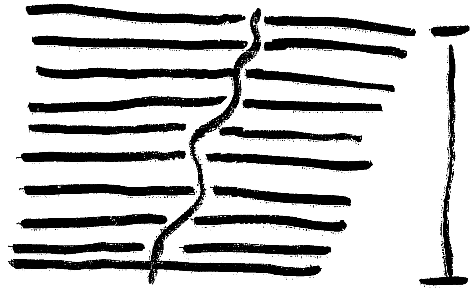


only w/ puncture

$(\mathcal{F},\mu)$ a measured foliation

α an essential simple closed curve

$$i(\alpha,(\mathcal{F},\mu)) = \inf_{\alpha_0 \simeq \alpha} \int_{\alpha_0} d\mu$$

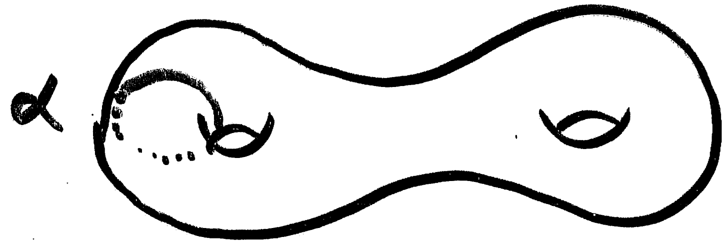


$$\mathcal{S}(S) = \{\text{ess. simple closed curves}\}/\mathrm{Homeo}_0(S)$$

$$\{\text{measured foliations}\} \to \mathbb{R}^{\mathcal{S}}$$

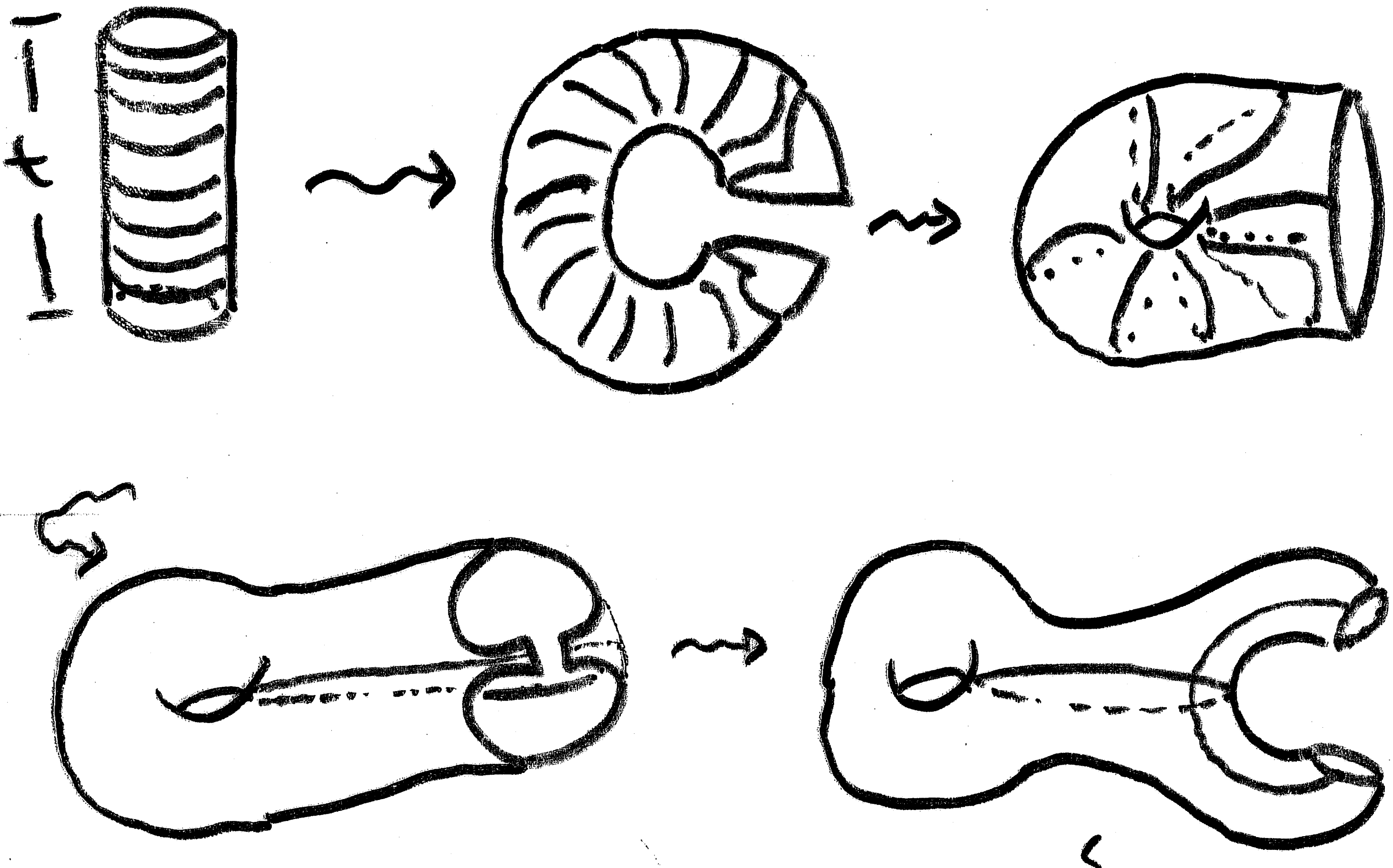
$$(\mathcal{F},\mu) \longmapsto \{i(\alpha,(\mathcal{F},\mu))\}_{\alpha\in\mathcal{S}}$$

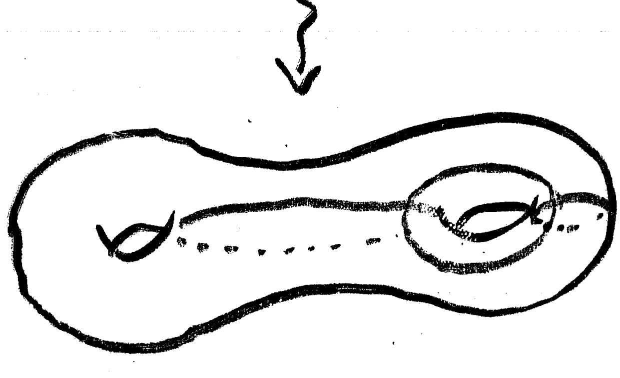
<u>Image is $\mathcal{MF}(S)$</u> $\cong \mathbb{R}^{2\xi(S)} \setminus \{0\}$



$\mathcal{S}(S) \times \mathbb{R}_+ \hookrightarrow \mathcal{MF}$

α an essential simple closed curve





$(\mathcal{F}_\alpha, \mu_\alpha)$

Geometric intersection number

$$i : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{Z}$$

extends continuously and homogeneously

$$i : \mathcal{MF} \times \mathcal{MF} \longrightarrow \mathbb{R}$$

$$\mathcal{PMF} = \mathcal{MF}/\mathbb{R}_+$$

Up to subsequence, degeneration of <u>hyperbolic</u> structures $\{\sigma_n\}_{n=1}^{\infty} \subset \mathcal{T}(S)$ is encoded by some $[\mu] \in \mathcal{PMF}$:

$$\frac{\text{Length}_{\sigma_n}(\alpha)}{\text{Length}_{\sigma_n}(\beta)} \longrightarrow \frac{i(\alpha, (\mathcal{F}, \mu))}{i(\beta, (\mathcal{F}, \mu))}$$

Teichmüller geodesics

$\sigma \in \mathcal{T}(S)$, q a quadratic differential

$$q = \{ \phi_\alpha : U_\alpha \to \mathbb{C} \}$$

holomorphic atlas away from a finite set of pts

st: • $\phi_\beta \circ \phi_\alpha^{-1} \Big|_{\text{component}} (z) = \pm z + \omega$, fixed $\omega \in \mathbb{C}$.

Pull-back Euclidean metric should have finite area. Want $\bar{S}$ = completion and have all cone angles $\geq 3\pi$, away from original punctures.

(σ_t, q_t):

$q_t = \{A_t \circ \phi_\alpha : U_\alpha \to \mathbb{C}\}$ $\quad A_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$

$t \longmapsto \sigma_t$ Teichmüller geodesic.

Can pull back $|dx|$: vertical measured foliation, v_{q_t}.

$$v_{q_t} = e^{t/2} v_q$$

"In q_t, v_q looks like its shrinking"

(Hubbard-Masur): $\forall\, \sigma \in \mathcal{T}(S)$

$$\left\{\begin{array}{c}\text{Non-zero } \sigma\text{-holomorphic} \\ q\text{-differentials}\end{array}\right\} \longrightarrow \mathcal{MF}$$

$$q \longmapsto v_q$$

is a homeomorphism.

ISOMETRIES

(Royden) usually, $\mathrm{Isom}^+(\mathcal{T}(S)) = \mathrm{Mod}(S)$.

Thurston Classification
(for Mod)

TRANSLATION LENGTH	REALIZED?	$\mathbb{H}^n$	$\mathcal{T}(S)$
$=0$	YES	Elliptic	FINITE ORDER
$=0$	NO	Parabolic	REDUCIBLE
>0	NO		
>0	YES	Hyperbolic	PSEUDO-ANOSOV

(Thurston):

$$\begin{array}{ccc} S & \longrightarrow & M \\ & & \downarrow \\ & & S^1 \end{array}$$

$$\Longrightarrow \quad \pi_1(S^1) \longrightarrow \langle \phi \rangle < \mathrm{Mod}(S)$$

$$M = S \times [0,1] / (x,1) \sim (\phi(x), 0)$$

$$M \cong \mathbb{H}^3 / \Gamma \ , \quad \Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$$

$$\Longleftrightarrow \quad \Gamma = \langle \phi \rangle, \ \phi \text{ pseudo-Anosov}.$$

SUBGROUPS

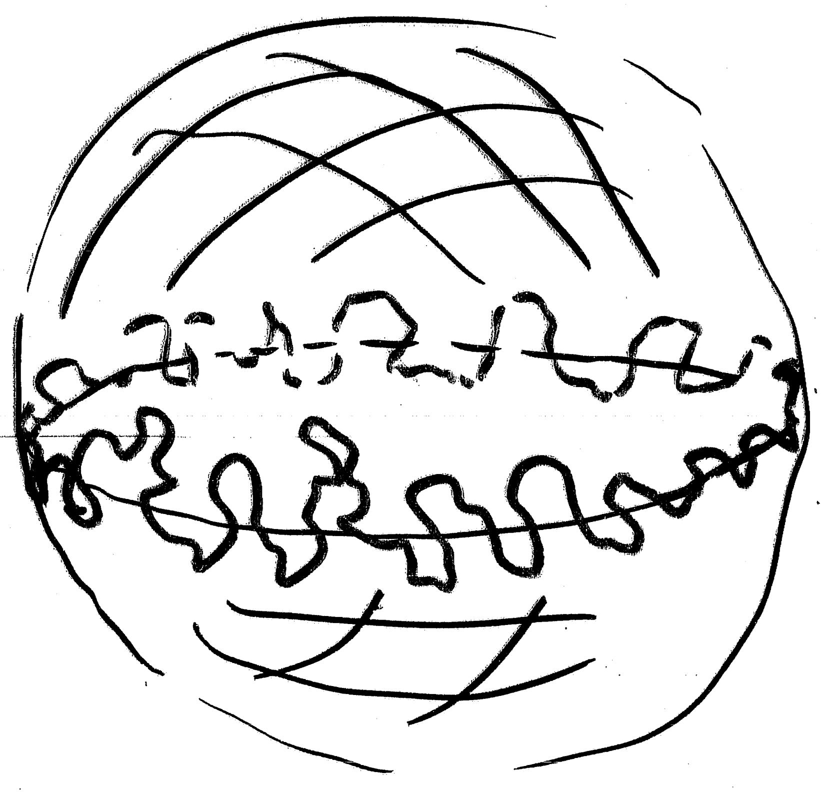
$G < \mathrm{Isom}^+(\mathbb{H}^n)$, Kleinian.

$G \curvearrowright S^{n-1}_\infty \Rightarrow$

$S^{n-1}_\infty = \Lambda_G \sqcup \Delta_G$

limit set

domain of discontinuity



(Masur, McCarthy - Papadopoulos)

$G < \mathrm{Mod}(S)$

[Assume G contains a p. Anosov]

$G \curvearrowright \mathcal{PMF}(S)$

$$\Lambda_G = \overline{\bigcup_{\substack{\phi \in G, \\ \text{p. Anosov}}} \mathrm{Fix}(\phi)} = \text{limit set}$$

IN GENERAL: $G \curvearrowright \mathcal{PMF} \setminus \Lambda_G$ NOT PROPERLY DISCONTINUOUS!

$Z\Lambda_G$ = "zero locus" of Λ_G w.r.t. i.

$\Delta_G = \mathcal{PMF} \setminus Z\Lambda_G$ = domain of discontinuity.

Q: Is Δ_G maximal? (sometimes yes...)

CONVEX COCOMPACTNESS

$G < \mathrm{Isom}^+(\mathbb{H}^n)$, Kleinian

T.F.A.E. (......................)

- $\mathrm{Hull}(G)/G$ compact
- $\mathbb{H}^n \cup \Delta_G/G$ compact
- Every limit point is conical

- $G \cdot x \subset \mathbb{H}^n$ is quasi-convex : $\exists A > 0$ s.t. $[hx, h'x] \subset N_A(G \cdot x)$
- $G \cdot x$ is f.g. quasi-isometrically embedded : $\exists K \geq 1, C \geq 0$

$$\frac{1}{K} d(h,h') - C \leq d(hx, h'x) \leq K\, d(h,h') + C$$

(Farb-Mosher)

$G < \mathrm{Mod}(S)$

$\Gamma_G = \pi_1 \begin{pmatrix} S\text{-bundle over } K(G,1) \text{ w/} \\ \text{monodromy } G \end{pmatrix}$

<u>Ex</u> $M = S \times [0,1] / (x,1) \sim (\phi(x), 0)$ $\quad |\phi| = \infty$

$\Rightarrow \pi_1(M) = \Gamma_{\langle \phi \rangle}$

S closed:

Γ_G δ-hyperbolic $\iff$ $G \cdot X \subset \mathcal{T}(S)$ is quasi-convex

($\Rightarrow$: Farb-Mosher ; $\Leftarrow$: Farb-Mosher, Hamenstädt)

G is convex cocompact if $G \cdot X$ is q. convex.

G convex cocompact $\Rightarrow$ G finitely generated purely pseudo-Anosov.

<u>Q</u> Is the converse true?

(see: Kent-L-Schleimer) for more...

G purely pseudo-Anosov $\Rightarrow$ Γ_G has no $\mathbb{Z} \oplus \mathbb{Z}$ (or $BS(p,q)$) subgroups.

<u>Q</u> (Gromov) If Γ has a finite $K(\Gamma,1)$, no $BS(p,q)$ subgroups, is Γ δ-hyperbolic?

(Farb-Mosher, Mosher)

PING-PONG $\Rightarrow$ lots of "Schottky" subgroups. — Free, convex cocompact.

Q. If $G < \mathrm{Mod}(S)$ is convex cocompact, is G virtually free?

Q. Does there exist $G < \mathrm{Mod}(S)$ convex cocompact w/ $G \cong \pi_1(S_{g,0})$? Does there exist $G < \mathrm{Mod}(S)$ purely pseudo Anosov w/ $G \cong \pi_1(S_{g,0})$?

(M. Kapovich) If $\exists\ G \cong \pi_1(S_{g,0}) < \mathrm{Mod}(S)$, convex cocompact, then $\Gamma_G \not\hookrightarrow \mathrm{Isom}(\mathbb{CH}^2)$:

THERE ARE NO $\mathbb{CH}^2$-surface bundles over surfaces.

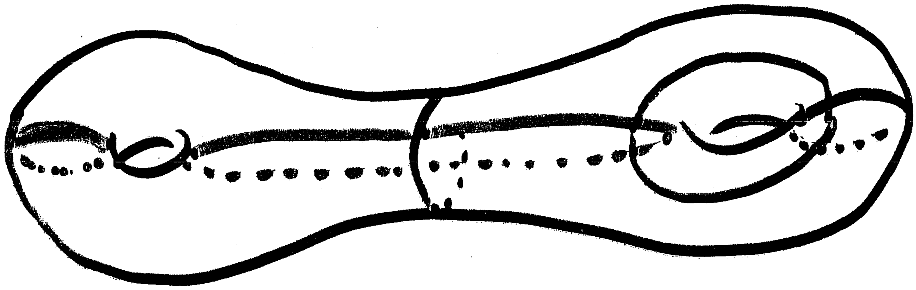
T.F.A.E. (Farb-Mosher, Hamenstädt, Kent-L

- $G < \mathrm{Mod}(S)$ convex cocompact
- Weak $\mathrm{Hull}(G)/G$ compact
- $\mathcal{T}(S) \cup \Delta_G / G$ compact
- all limit points are conical
- $G \cdot \alpha < \mathcal{C}(S)$ is quasi isometrically embedded

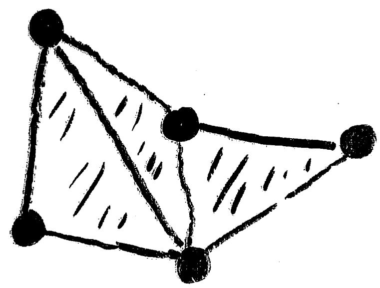
$\mathcal{C}(S)$ = complex of curves of S ....

Another day!

$\mathcal{C}(S)$



small piece



- locally infinite
- δ-hyperbolic (Masur-Minsky)
- infinite diameter (Kobayashi)