# Cornell Topology Festival Panel Discussion

May 2007

The 2007 Topology Festival had a concentration of speakers in *group theory and low dimensional topology*. Two of the speakers gave introductory workshops on the subject — Chris Leininger spoke on *Mapping Class Groups Versus Kleinian Groups* and Tim Riley on *Open Problems in Low-Dimensional Topology and Group Theory*.

The speakers featured in a panel discussion on Wednesday 23rd May in which they presented some recent results, not their own, that had caught their attention. Peter Kahn, one of the Cornell organisers, acted as moderator.

**Danny Calegari** (*California Institute of Technology*)

In the 2004 paper by Kontsevich and Soibleman, entitled "Affine structures and non-archimedean analytic spaces," the authors obtain a faithful action of some arithmetic lattice in SO(1,18) on the 2-sphere by integral piecewise linear homeomorphisms. It would be nice to have a more transparent construction of such an action.

Is the group $Homeo(D^2, \partial D^2)$ of homeomorphisms of a disk which fix the boundary left-orderable? The guess is that it is not.

**Ralph Cohen** (*Stanford*)

If $M$ is a smooth $n$-dimensional manifold, then the cotangent bundle $T^*M$ has a canonical symplectic structure. Many new invariants have been introduced relating to this symplectic structure, such as Witten invariants, field theory invariants, etc. What can these invariants of the cotangent bundle tell us about the smooth topology of M?

Viterbo showed that the Symplectic Floer homology $HF_*(T^*M)$ of the cotangent bundle corresponds to the homology $H_*(LM)$ of the loop space of $M$, where closed geodesics in $LM$ correspond to periodic orbits of the Hamiltonian flow. Here Gromov-Witten invariants should correspond to intersection invariants of $LM$, such as those introduced by Sullivan and others.

This has been studied by K. Fukaya, and by K. Cieliebak and J. Latschev.

A question inspired by Mark Feighn's presentation: There is a map from the monoid of homotopy equivalences $HE(\vee S^1)$ of a wedge of $n$ circles to the group $Aut(F_n)$ of automorphisms of a free group. Does this map have a section?

Thurston then asked: Can you realize $Aut(F_n)$ or $Out(F_n)$ as a group of diffeomorphisms of any space? Or, for surfaces, is there any finite-dimensional manifold on which the mapping class group can be realized as a group of diffeomorphisms?

**Cornelia Druţu** (*Université des Sciences et Technologies de Lille I*) on Property $(T)_p$.

Recently Chatterji, Druţu and Haglund [7] proved the following for locally compact, second–countable groups $G$.

1. If there exists $p > 0$, such that every action of $G$ by affine isometries on any $L^p(X, \mu)$ has bounded orbits (or equivalently, in the case $p > 1$, has a fixed point), then $G$ has Property $(T)$.

2. If $G$ is a-$(T)$-menable (that is, has the Haagerup property), then for all $p > 0$, $G$ has a proper affine isometric action on some $L^p$ space.

Moreover, by results of Delorme [9], Guichardet [11] and Akemann–Walter [1], partial converses hold:

1′. If $G$ has Property $(T)$, then for all $p \in (0, 2]$, any action of $G$ by affine isometries on any $L^p(X, \mu)$ has bounded orbits.

2′. If $G$ has a proper affine isometric action on some $L^p$ space where $p \in (0, 2]$, then $G$ is a-$(T)$-menable.

These partial converses are false for $p > 2$ on account of a result of G. Yu [17]: every non-elementary hyperbolic group $G$ acts properly on $L^p(G \times G)$ for sufficiently large $p$.

A group $G$ has Property $(T)_p$ if every action of $G$ on every $L^p$-space has bounded orbits. It is a-$(T)_p$-menable if it has a proper action on some $L^p$ space. Note that by the above, properties $(T)$ and $(T)_p$ agree for all $p \in (0, 2]$.

Fisher and Margulis (unpublished, see [3]) showed that if a group $G$ has $(T)$ then there is some $\epsilon > 0$ (depending on $G$) such that $G$ has $(T)_p$ for all $p \in (0, 2+\epsilon)$.

Druţu asked what one can say for non-elementary hyperbolic groups $G$ having property $(T)$ as far as $(T)$ and $(T)_p$ are concerned? Note that these groups are a-$(T)_p$-menable for all sufficiently large $p$ by the result of Yu, and some non-elementary hyperbolic groups $(T)$. Is there some $p_0$ such that $G$ is $(T)_p$ for all $p < p_0$ and is a-$(T)_p$-menable for all $p > p_0$?

Also Druţu asked whether something similar can be said about $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{H}})$, or $\mathrm{Isom}(\mathbb{H}^2_{\mathrm{Cay}})$ and their lattices? Note that $G = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{H}}) = \mathrm{Sp}(n, 1)$ does have $(T)$ and de Cornulier, Tessea and Valette [8] proved that $G$ acts properly on $L^p(G)$ for $p > 4n + 2$. (The Hausdorff dimension of the boundary is $4n + 2$.)

Consider the set $R$ of reduced words of length 3 in $F_m$. (There are $2m(2m-1)^2$.) Fix $\beta \geq 0$ and let $S$ be a subset of $2m(2m-1)^{2\beta}$ elements of $R$ chosen uniformly at random. Żuk [18] proved that if $\beta < 1/2$ then the probability that $F_m/S$ is non-elementary hyperbolic tends to 1 as $m \to \infty$, and if $\beta > 1/3$ then the probability that $F_m/S$ has $(T)$ tends to 1 as $m \to \infty$. Druţu asked whether there is some similar result for $(T)_p$ when $p > 2$?

**Alex Eskin** (University of Chicago) *on the probability of picking a pseudo-Anosov in the mapping class group.*

By the Thurston classification one divides the elements in the mapping class group $\Gamma$ into periodic, reducible and pseudo-Anosov elements. It is said, that "almost all elements" are pseudo-Anosov. However, one has to make precise how the elements are picked from the mapping class group. Here are two different ways, to define this:

A) Rivin and Maher show that along a random walk one hits a pseudo-Anosov element with probability 1 (see [14] and [12]). Furthermore the translation length goes linearly to infinity as the length of the walk goes to infinity.

B) Fix a generating set of the mapping class group and pick $g$ randomly as a word in the generators. The statement has apparently not been proved for this setting so far. It is expected to be a difficult problem, since $\Gamma$ is not a hyperbolic group. The answer should probably not depend on the generating set.

Similar questions can be asked for other groups. The statement in setting A) true for non-elementary subgroups of the mapping class group such as the Torelli group (see [12]). Furthermore, it is natural to study these problems for the outer automorphism group and $\mathrm{SL}_n(\mathbb{Z})$. In particular it is an interesting question whether a randomly picked matrix in $\mathrm{SL}_n(\mathbb{Z})$ has an irreducible characteristic polynomial. This is shown for setting A) in Rivin's work.

**M. Feighn** (*Rutgers*).

If $M$ is a closed oriented surface there is a natural homomorphism $Pr$ from the group $Homeo(M)$ of orientation-preserving homeomorphisms onto the mapping class group $Mod(M)$. A theorem of Morita says that the analogous map from

*Diff*($M$) to *Mod*($M$) does not split: the group *Mod*($M$) cannot be realized as a group of diffeomorphisms of $M$, but the proof does not work for *Homeo*($M$). In a recent paper to appear in *Inventiones*, Vladimir Markovic proves that *Pr* does not split if the genus of the surface is greater than 5.

It's a proof by contradiction. He assumes there's a lift. In that case, the lift of a Dehn twist has to look a lot like a Dehn twist. A key ingredient is to associate an $f$-invariant (upper semi-continuous) decomposition of $M$ to each $f \in Homeo(M)$. Markovic shows that this decomposition has to be non-trivial for certain $f$ in the image of the lift.

The proof relies on arguments with torsion elements, and the question is still open for torsion-free subgroups of finite index. Markovich thinks that his techniques might extend to genus greater than 2.

Thurston asked whether there was a nice quasi-morphism backwards.

**I. Kapovich** (*Urbana Champaign*) on a paper of S. Schleimer, *Polynomial-Time Word Problems* [15].

The main result of this paper is that the word problems for $\mathrm{Aut}(F_n)$ and for $F_n \rtimes_\Phi \mathbb{Z}$ (where $\Phi \in \mathrm{Aut}(F_n)$) are solvable in polynomial time.

A crude solution to the word problem in

$$F_n \rtimes_\Phi \mathbb{Z} \;=\; \langle a_1, \ldots, a_n, t \mid t^{-1} a_i t = \Phi(a_i) \rangle$$

is to proceed as follows. First check that the letters $t^{\pm 1}$ in the given word have exponent sum 0 and then collect the $t^{\pm 1}$ at one end of the word — however, with this approach, the word can grow exponentially long and the algorithm can take exponential time. There are similar inefficient solutions to the word problem in $\mathrm{Aut}(F_n)$.

However techniques from Computer Science, specifically *straight-line programs* (a form of compression), can be used in both cases to turn this into a deterministic polynomial time algorithm for the word problem.

One can speculate that there may be connections with the theory of Makanin–Razborov diagrams.

**Chris Leininger** (University of Illinois at Urbana-Champaign) *on work of David Dumas and Richard P. Kent* [10].

The main result of this work is that Bers slices are not algebraic.
Recall that Bers slices are defined as follows: Let $S$ be a closed connected oriented

hyperbolic surface and let $\chi_{\mathbb{C}}(S)$ be its $\mathrm{SL}_2(\mathbb{C})$-character variety. Then $\chi_{\mathbb{C}}(S)$ contains the space $\mathrm{QF}(S)$ of quasifuchsian groups. By the Simultaneous Uniformization Theorem of Bers $\mathrm{QF}(S)$ is parametrized by $\mathrm{T}(S) \times \mathrm{T}(\overline{S})$. For $Y \in T(\overline{S})$, we look at

$$B_Y = T(S) \times \{Y\} \subseteq T(S) \times T(\overline{S}) = \mathrm{QF}(S) \hookrightarrow \chi_{\mathbb{C}}(S),$$

which is an analytic $3g-3$ dimensional subvariety of $\chi_{\mathbb{C}}(S)$. $B_Y$ is called *Bers slice*. In [10] it is shown that a Bers slice $B_Y$ is not contained in any $3g-3$ dimensional algebraic subvariety $V \subseteq \chi_{\mathbb{C}}(S)$. Thus the dimension of its Zariski closure is at least $3g-2$. One corollary of this result is that skinning maps are never constant.

- *Question:* What is the dimension of the Zariski closure of a Bers slice $B_Y$? It actually seems natural to wonder, whether $B_Y$ is in fact Zariski-dense in $\chi_{\mathbb{C}}(S)$.
- *Question (Serge Cantat):* Could $B_Y$ be a leaf of an algebraic foliation?

**Tim Riley** (*Cornell University*) on work of M.R. Bridson, J. Howie, C.F. Miller III, and H. Short on *Subgroups of Direct Products of Limit Groups* [5].

Bridson, Howie, Miller and Short proved a conjecture of Sela stating that if $\Gamma_1, \dots, \Gamma_n$ are *limit groups* and $S \leq \Gamma_1 \times \cdots \times \Gamma_n$ enjoys the finiteness property $\mathrm{FP}_n$, then there exists a finite index subgroup $T$ of $S$ that is a direct product of at most $n$ limit groups.

This adds weight to the idea that Sela's limit groups are a natural extension of the class of free groups. [Limit groups can be defined in a number of different ways: a limit group is a limit of a sequence of *marked free groups* in the space $\mathcal{G}_k$ of *marked groups* (the elements of $\mathcal{G}_k$ are groups equipped with a generating $k$-tuple, and two are considered to be close when their Cayley graphs agree on a large ball); limit groups are the finitely generated fully residually free groups; also limit groups have a characterisation in terms of first order logic – they are the $\exists$-*free groups*.]

As background we recall that Baumslag and Roseblade [4] proved the result in the case where $n = 2$ and $\Gamma_1$ and $\Gamma_2$ are free. And Bridson, Howie, Miller and Short [6] extended this to arbitrary $n$ with $\Gamma_1, \dots, \Gamma_n$ surface groups (including free groups).

Recall that the Bieri–Stallings group $\mathrm{BS}_n$ is the kernel of the map $(F_2)^k \to \mathbb{Z}$ in which, expressing each $F_2$ as $\langle a_i, b_i \rangle$, every $a_i$ and $b_i$ is mapped to 1. $\mathrm{BS}_n$ is of type $\mathrm{FP}_{n-1}$ but not $\mathrm{FP}_n$. So the result says that if one imposes the appropriate finiteness property on $S$ to rule out the Bieri–Stallings examples, the only $S$ that remain are the most obvious ones.

The authors make an intriguing but unexplained suggestion that their results might have significance for the study of Kähler Geometry. They say that knowing whether every finitely presentable subdirect product of a product of surface groups satisfies a polynomial isoperimetric inequality would be useful for understanding Kähler groups.

**Juan Souto** (University of Chicago): *A collection of questions about spines.*

One may start with the following question:

- *Question:* What is a good spine for $\mathrm{T}_g$?

More generally, consider $\mathrm{SL}_n(\mathbb{Z})$-invariant spines in $\mathrm{SO}_n(\mathbb{R})\backslash\mathrm{SL}_n(\mathbb{R})$. A good spine was found by Soulé and Ash (see [2] and [16]) called a *well-rounded retract*. It naturally carries the structure of a cell complex.

- *Question*: Is the well-rounded retract minimal or does it contain another $\mathrm{SL}_n(\mathbb{Z})$-invariant spine?

By a theorem of McMullen (see [13]), a periodic maximal flat $F$ in $\mathrm{SO}_n(\mathbb{R})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ is not disjoint from the well-rounded retract $X$.

- *Question:* Is this statement true for arbitrary deformation retracts?

In fact, one can show using the arguments in McMullen's proof that $F$ cannot be homotoped to $\infty$.

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