Cornell University

The geometry of discs spanning loops in groups and spaces

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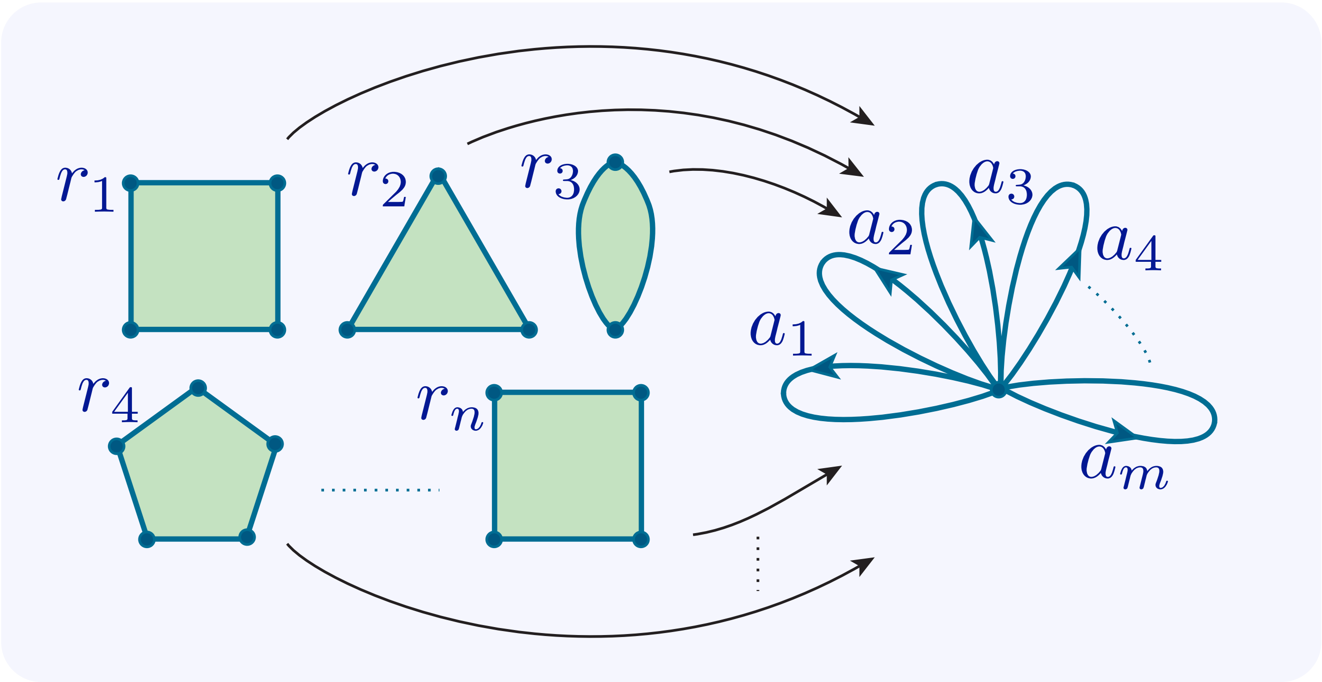
$\mathcal{P} = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$ a finite presentation of a group Γ

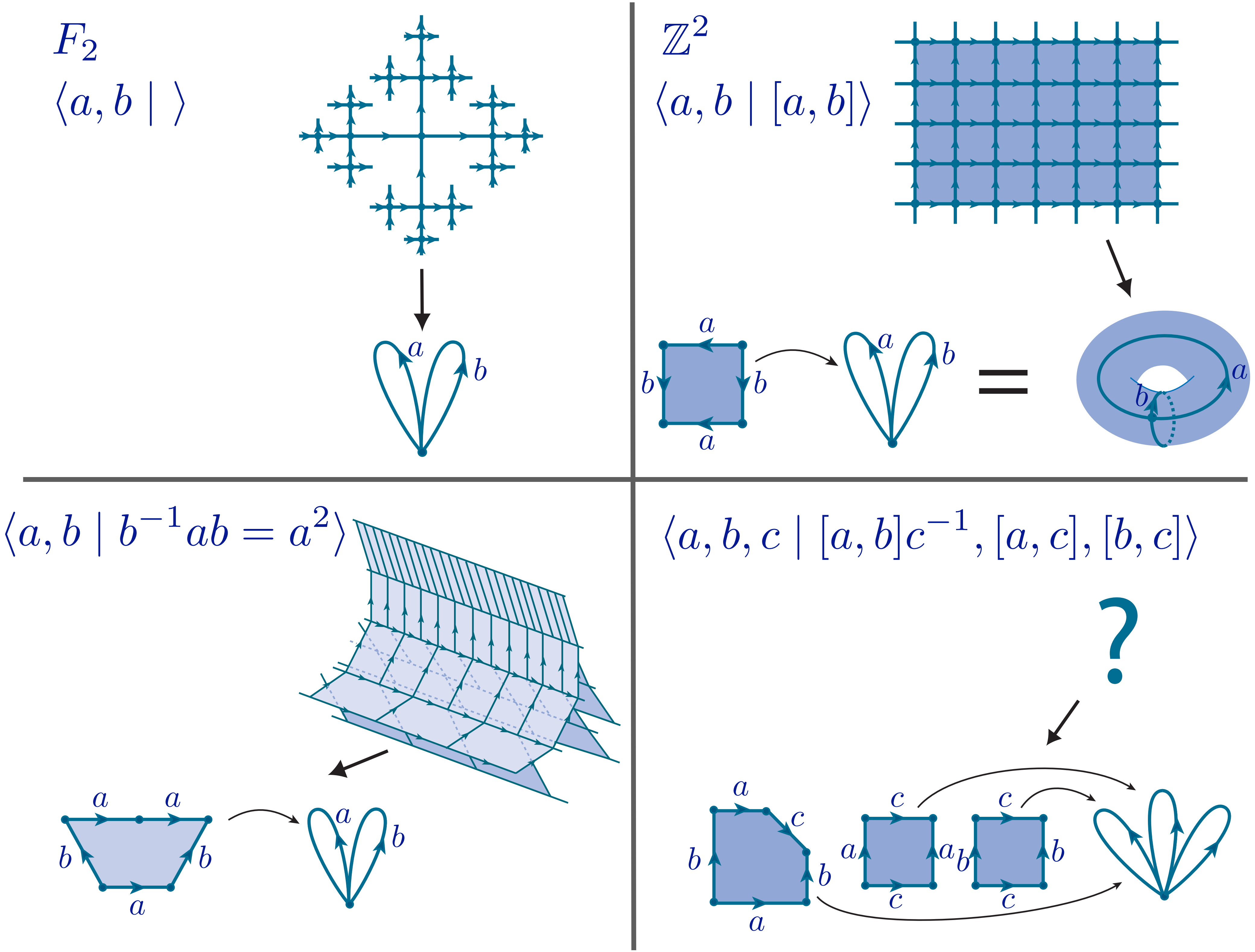
The presentation 2-complex of $\mathcal{P}$:

$K =$ 

$\pi_1(K) = \Gamma$ (Seifert–van Kampen Theorem)

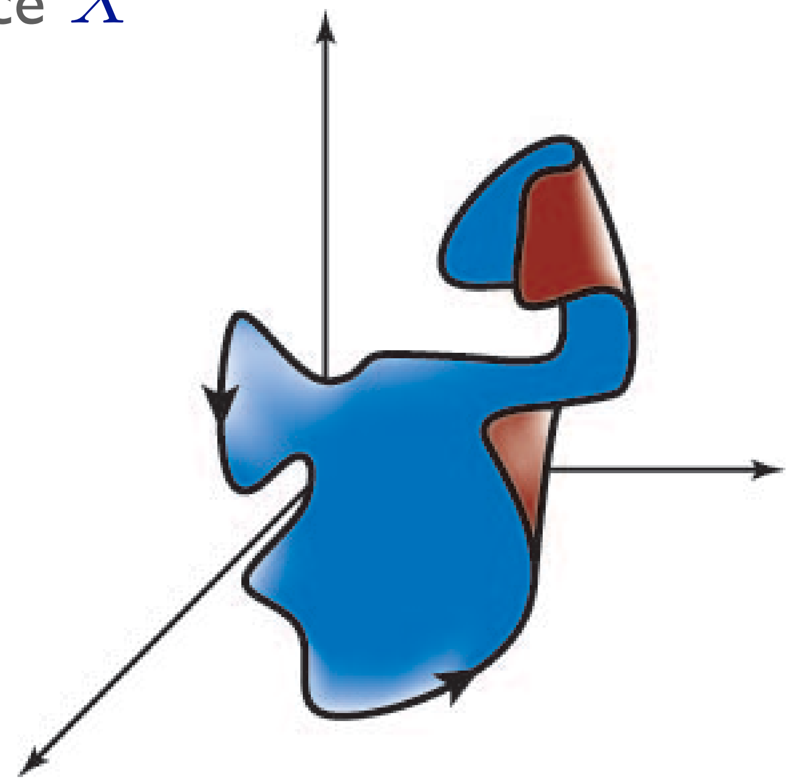
The universal cover $\widetilde{K}$ is the *Cayley 2-complex* of $\mathcal{P}$.

Its 1-skeleton $\widetilde{K}^{(1)}$ is the *Cayley graph* of $\mathcal{P}$.


F_2
$\langle a, b \mid \ \rangle$
a
b
$\mathbb{Z}^2$
$\langle a, b \mid [a, b] \rangle$
a
b
b
a
a
b
=
a
b
$\langle a, b \mid b^{-1}ab = a^2 \rangle$
a
a
b
b
a
a
b
$\langle a, b, c \mid [a, b]c^{-1}, [a, c], [b, c] \rangle$
?
a
c
b
b
a
c
a
a
c
c
b
b
c

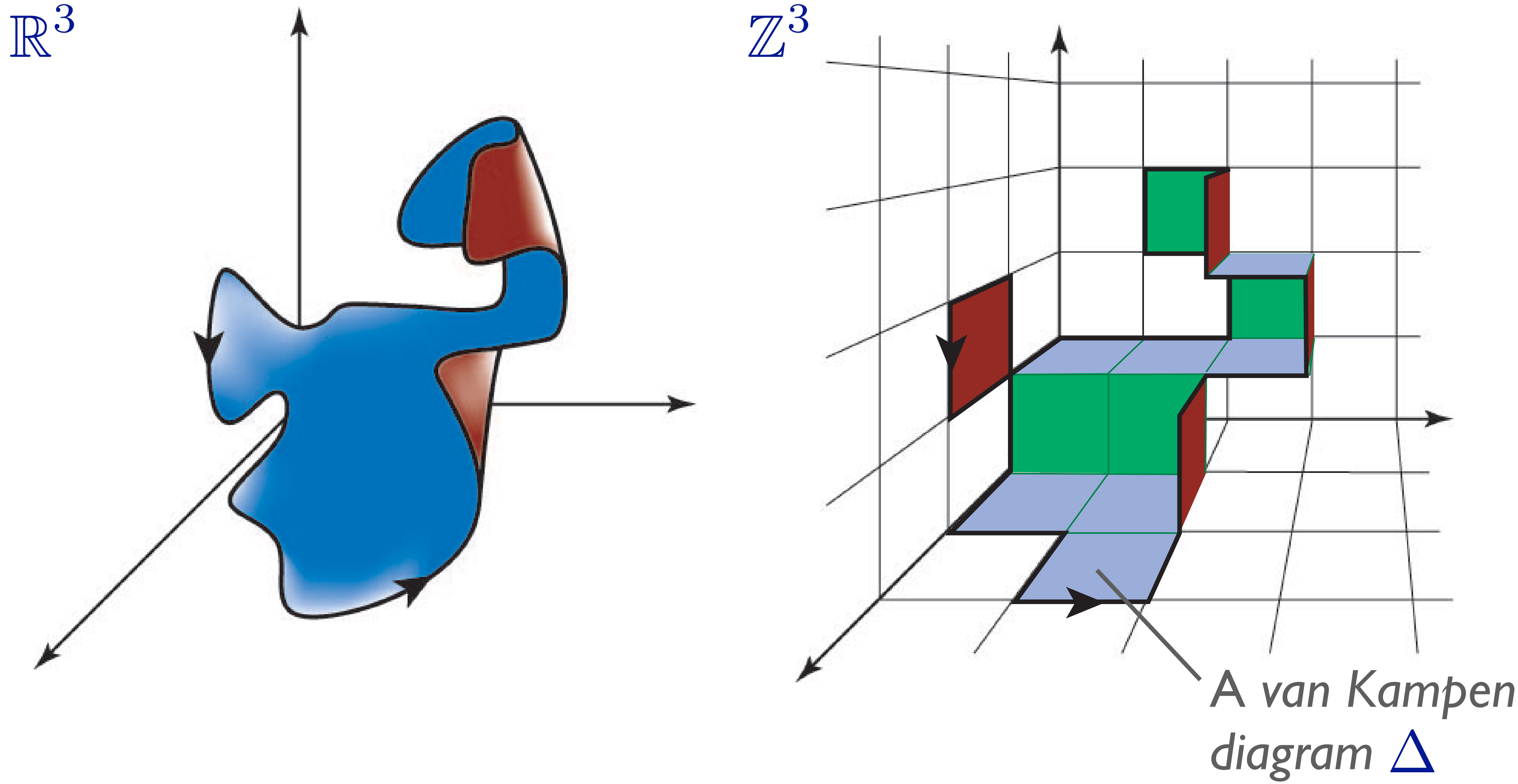
ρ a loop in a simply connected space X

$\mathrm{Area}(\rho)$ is the infimum of the areas of discs spanning ρ.



$\mathrm{Area}_X : [0, \infty) \to [0, \infty]$ is defined by

$$\mathrm{Area}_X(l) \; = \; \sup\{\mathrm{Area}(\rho) \mid \ell(\rho) \leq l\}.$$



$\text{Area}(\Delta) = \#$ 2-cells

For an edge-loop ρ in the Cayley 2-complex of a finite presentation $\mathcal{P}$, $\text{Area}(\rho)$ is the minimum of $\text{Area}(\Delta)$ over all van Kampen diagrams spanning ρ.

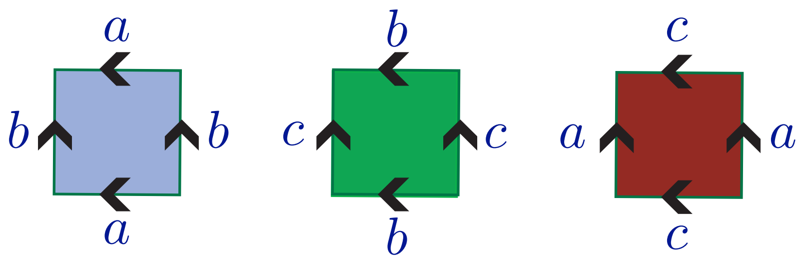
The *Dehn function* $\mathrm{Area}_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ of a finite presentation $\mathcal{P}$ with Cayley 2-complex $\widetilde{K}$ is

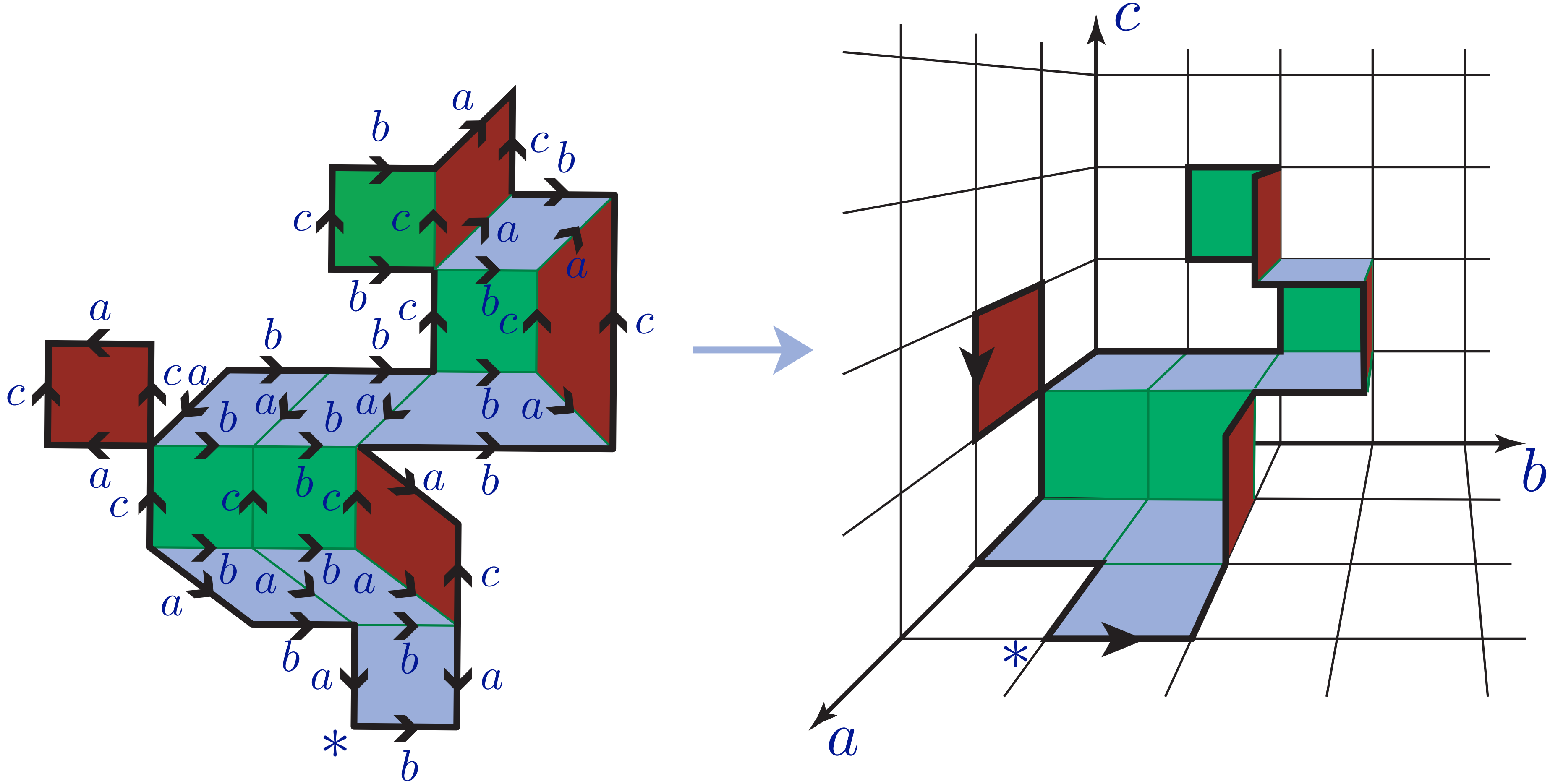
$$\mathrm{Area}_{\mathcal{P}}(n) \ = \ \max\{\mathrm{Area}(\rho) \mid \text{edge-loops } \rho \text{ in } \widetilde{K} \text{ with } \ell(\rho) \leq n\}.$$

The Filling Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group of a closed Riemannian manifold M then

$$\mathrm{Area}_{\mathcal{P}} \ \simeq \ \mathrm{Area}_{\widetilde{M}}.$$

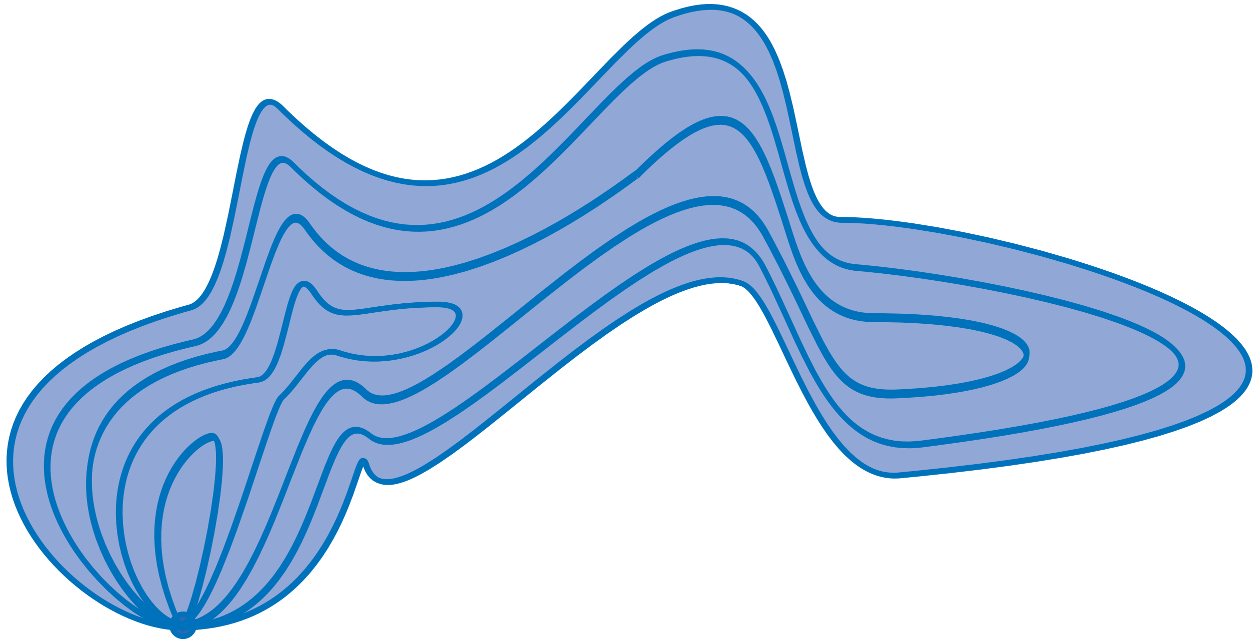
$\mathbb{Z}^3$ $\quad\quad$ $\langle a, b, c \mid [a, b], [b, c], [c, a] \rangle$





Filling Length

ρ a loop in a simply connected space X

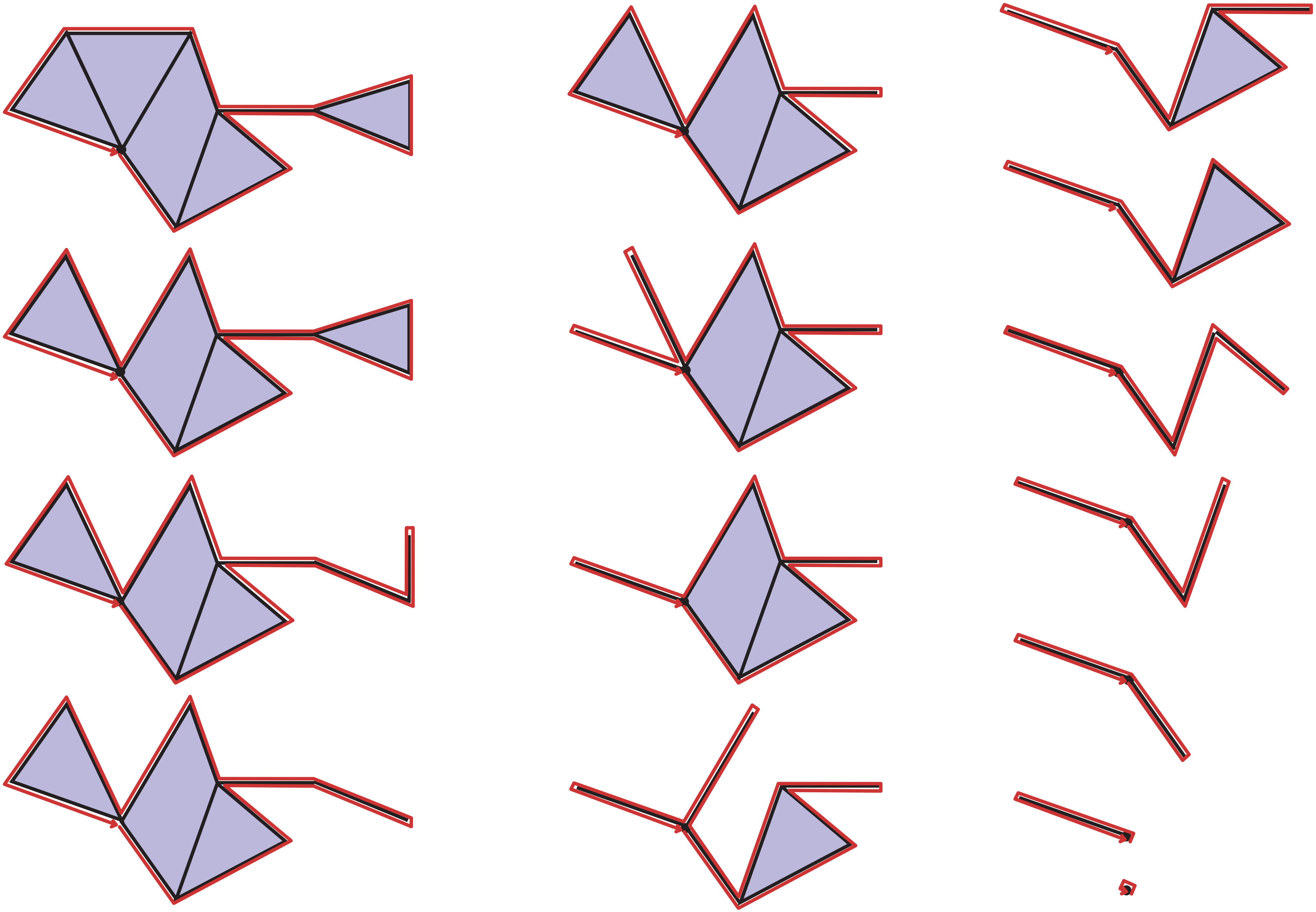


$$\mathrm{FL}(\rho) \ = \ \inf \left\{ L \ \middle| \ \begin{array}{l} \exists \text{ a (based) null-homotopy of } \rho \\ \text{through loops of length } \leq L \end{array} \right\}$$

The *filling length function* $\mathrm{FL}_X : [0, \infty) \to [0, \infty]$ is defined by

$$\mathrm{FL}_X(l) \ = \ \sup\{\mathrm{FL}(\rho) \mid \ell(\rho) \leq l\}.$$

A combinatorial null-homotopy across a singular combinatorial 2-disc



The *filling length* $\mathrm{FL}(\Delta)$ of a (singular) combinatorial 2-disc Δ is the minimum L such that $\partial\Delta$ can be combinatorially null-homotoped across Δ.

For an edge-loop ρ in the Cayley 2-complex of a finite presentation $\mathcal{P}$, $\mathrm{FL}(\rho)$ is the minimum of $\mathrm{FL}(\Delta)$ over all van Kampen diagrams spanning ρ.

The *filling length function* $\mathrm{FL}_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{P}$ is

$$\mathrm{FL}_{\mathcal{P}}(n) \; = \; \max\{\mathrm{FL}(\rho) \mid \text{edge-loops } \rho \text{ in } \widetilde{K} \text{ with } \ell(\rho) \leq n\}.$$

Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group of a closed Riemannian manifold M then

$$\mathrm{FL}_{\mathcal{P}} \simeq \mathrm{FL}_{\widetilde{M}} \, .$$

The Word Problem

$\mathcal{P} = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$ a finite presentation of a group Γ

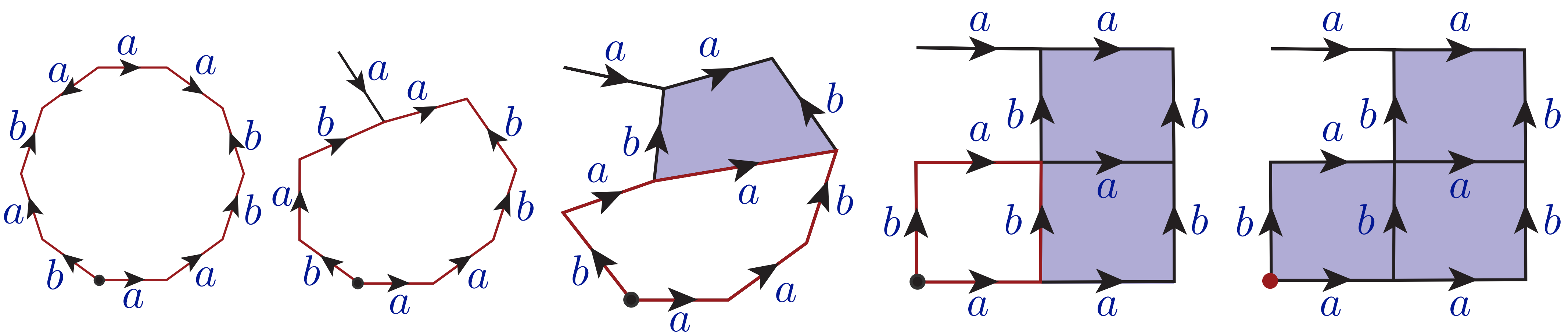
M. Dehn: Does there exist a systematic procedure (an algorithm) which on input a word w declares whether or not w represents 1 in Γ?

Naïve attack: exhaustively apply relations to w in the hope of obtaining the empty word. "The *Dehn Proof System*."

Example

$\langle a, b \mid [a, b] \rangle$

a	a	b	b	a^{-1}	a^{-1}	a	b^{-1}	a^{-1}	b^{-1}
	a	a	b	b	a^{-1}	b^{-1}	a^{-1}	b^{-1}	
		a	a	b	a^{-1}	a^{-1}	b^{-1}		
			a	b	a^{-1}	b^{-1}			



Dehn function = (non-deterministic) TIME

Filling length function = SPACE

—of the Dehn Proof System

Mantra:

Filling functions $\longleftrightarrow$ algorithmic complexity measures

Theorem (Sapir–Birget–Rips). If $f(n)$ is the time function of a non-deterministic Turing machine and f^4 is super-additive, then there is a finite presentation for which $\mathrm{Area}(n) \simeq f^4(n)$.

Other filling functions

The *diameter* of a singular combinatorial 2-disc Δ is the maximum distance between two of its vertices in the combinatorial metric on $\Delta^{(1)}$.

The *gallery length* of Δ is the maximum distance between two vertices in the graph dual to $\Delta^{(1)}$.

Define the (*intrinsic*) *diameter* $\mathrm{IDiam}(n)$ and *gallery length* $\mathrm{GL}(n)$ filling functions of a finite presentation as for $\mathrm{Area}(n)$.

The *extrinsic diameter* of a van Kampen diagram $\Delta \subset \widetilde{K}$ is the maximum distance between two of its vertices in the combinatorial metric on $\widetilde{K}^{(1)}$.

Thereby define the extrinsic diameter filling function $\mathrm{EDiam}(n)$.

What do these mean computationally?

Relationships between filling functions

The SPACE–TIME bound

Theorem. Given a finite presentation, $\exists C > 0,\ \forall n$,

$$\mathrm{Area}(n) \leq C^{\mathrm{FL}(n)}.$$

The Double Exponential Theorem

Theorem (D.E.Cohen)**.** Given a finite presentation,

$$\mathcal{P} = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_s \rangle,$$

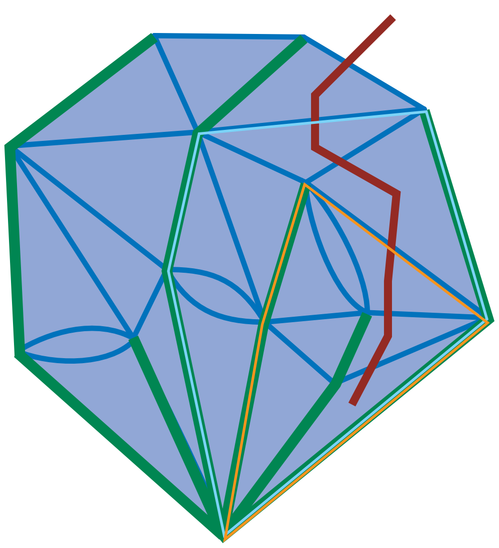
$\exists C > 0, \ \forall n,$

$$\text{Area}(n) \leq nC^{C^{\text{IDiam}(n)}}.$$

Proof (Gersten, R.)**.**

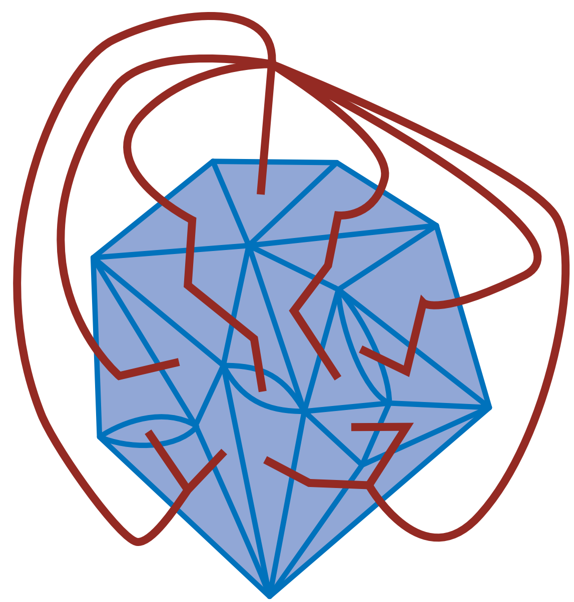
$\text{GL}(n) \leq 2{C_1}^{1 + 2\,\text{IDiam}(n)}$

where $C_1 = 1 + 2m$.



$\text{Area}(n) \leq n{C_2}^{\text{GL}(n)}$

where $C_2 = \max_j \ell(r_j)$.



J.R. Stallings: Given a finite presentation, does there always exist $C > 0$ such that $\forall n$

$$\mathrm{Area}(n) \leq C^{\mathrm{IDiam}(n)} \text{ ?}$$

M. Gromov: Given a finite presentation, does there always exist $C > 0$ such that $\forall n$

$$\mathrm{FL}(n) \leq C\,\mathrm{IDiam}(n) \text{ ?}$$

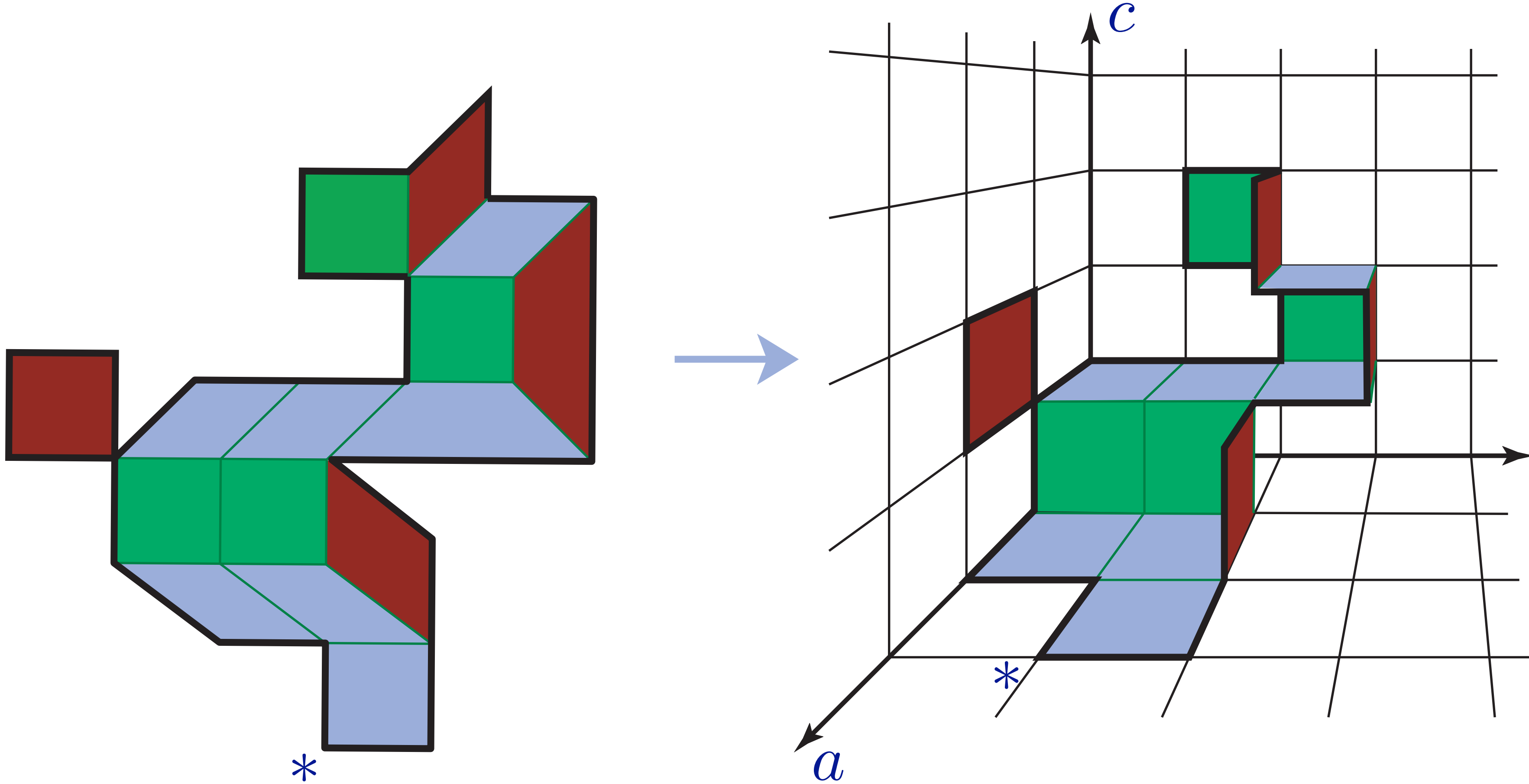
S. Gersten, R.: Is

$$\mathrm{FL}(n) \simeq \mathrm{GL}(n)$$

for every finite presentation?

Intrinsic versus extrinsic diameter

Is measuring diameter of van Kampen diagrams extrinsically qualitatively different to measuring it intrinsically?



Are there finite presentations for which $\mathrm{EDiam}(n) \not\simeq \mathrm{IDiam}(n)$?

Theorem (Bridson, R.). Yes!

For all $\alpha > 0$, there is a finite presentation with

$$n^{\alpha}\,\mathrm{EDiam(n)} \;\preceq\; \mathrm{IDiam}(n).$$

Our family of groups:

$$\Psi_{k,m} = \Phi_k *_{\langle t \rangle} \Gamma_m$$

amalgamated along an infinite cyclic subgroup $\langle t \rangle$.

Presentation of Γ_m

generators $a_1, \ldots, a_m, \sigma, t, \tau, T$

relations $\sigma^{-1} a_m \sigma = a_m;\ \forall\, i < m,\ \sigma^{-1} a_i \sigma = a_i a_{i+1}$
$\forall\, j,\ [t, a_j] = 1, [t, T], [\tau, T],$
$[\tau, a_m t],\ \forall\, i < m,\ [\tau, a_i]$

Presentation of Φ_k

generators
$s_1, \ldots, s_k, f, g \quad \hat{s}_1, \ldots, \hat{s}_k, \hat{f}, \hat{g} \quad b, t$

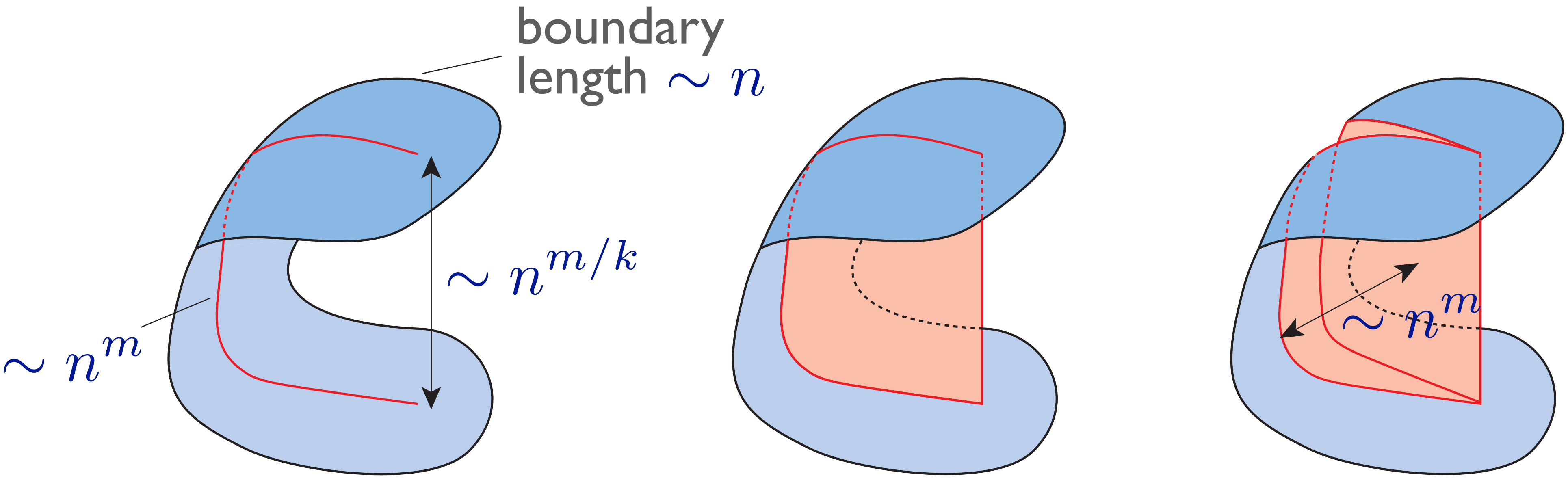
relations
$t^{-1} b s_k = b^3,\ s_k{}^{-1} b t = b^3,\ \hat{s}_k{}^{-1} b \hat{s}_k = b^3$

$\forall i < k,\quad f^{-1} s_k f = s_k,\ f^{-1} s_i f = s_i s_{i+1},\ \hat{f}^{-1} \hat{s}_k \hat{f} = \hat{s}_k,\ \hat{f}^{-1} \hat{s}_i \hat{f} = \hat{s}_i \hat{s}_{i+1}$

$g^{-1} s_k g = s_k,\ g^{-1} s_{k-1} g = s_{k-1},\ \hat{g}^{-1} \hat{s}_k \hat{g} = \hat{s}_k,\ \hat{g}^{-1} \hat{s}_{k-1} \hat{g} = \hat{s}_{k-1}$

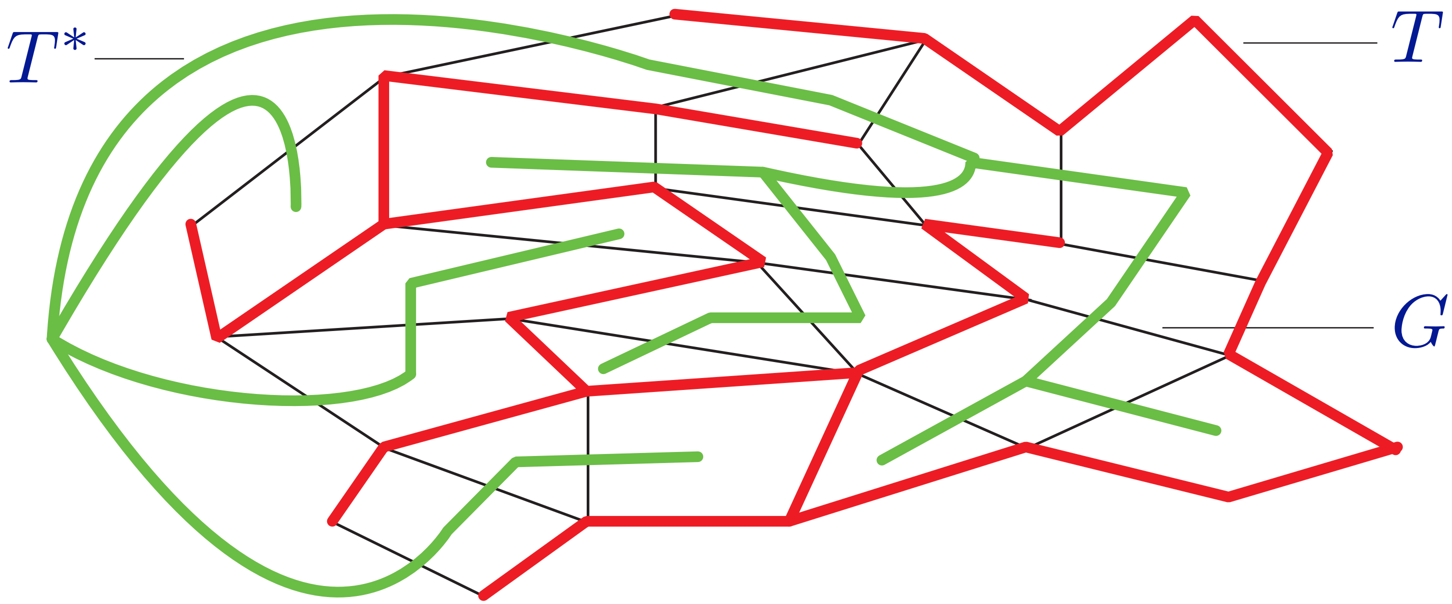
$\forall i < k-1,\ g^{-1} s_i g = s_i s_{i+1},\quad \hat{g}^{-1} \hat{s}_i \hat{g} = \hat{s}_i \hat{s}_{i+1}$

$\forall i \neq j,\quad [s_i, s_j] = 1,\quad [\hat{s}_i, \hat{s}_j] = 1$



boundary
length $\sim n$
$\sim n^{m/k}$
$\sim n^m$
$\sim n^m$

Dual trees in planar graphs



Question (Gersten–R.). Does there exist $K > 0$ such that for all finite planar connected graphs G, there is a spanning tree T in G with

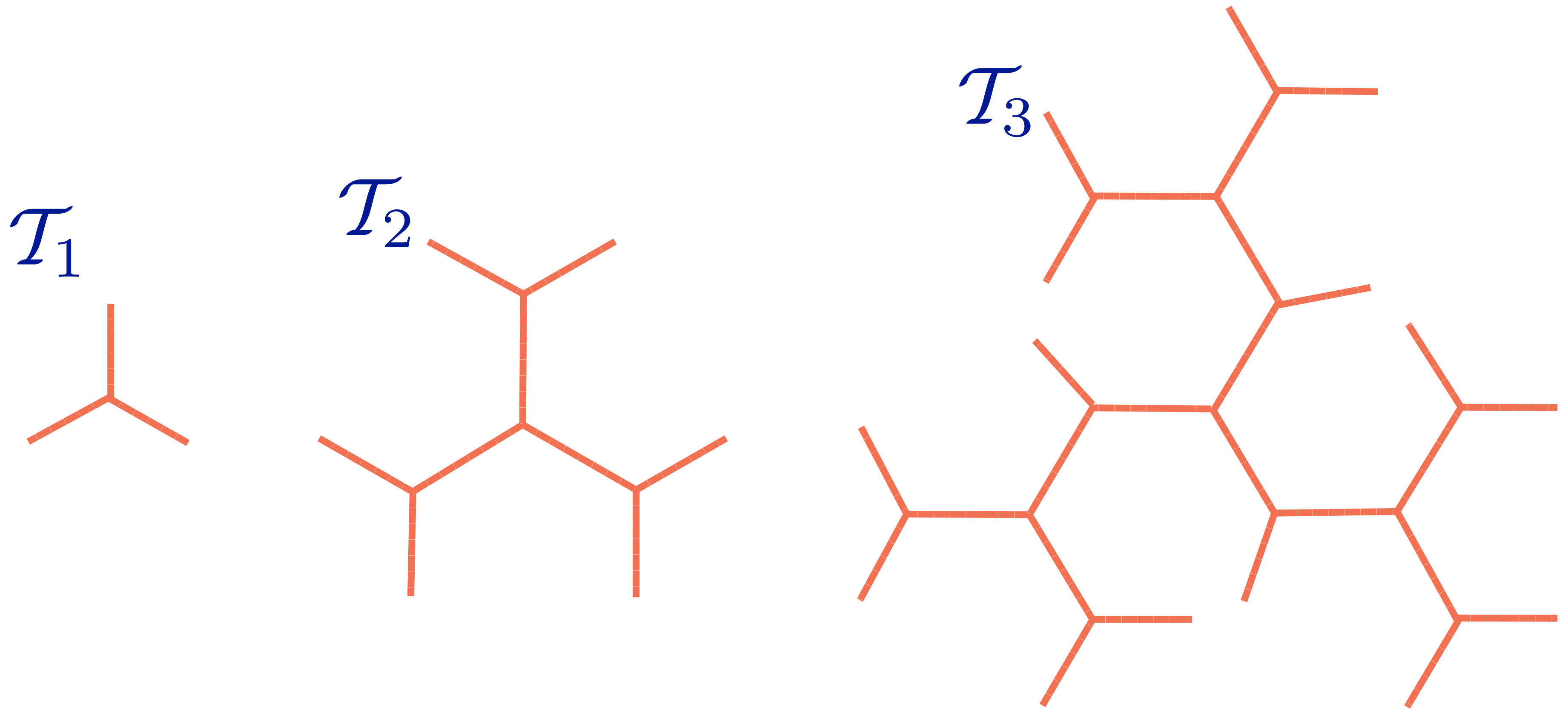
$$\mathrm{Diam}(T) \leq K \, \mathrm{Diam}(G) \text{ and}$$

$$\mathrm{Diam}(T^*) \leq K \, \mathrm{Diam}(G^*) \text{ ?}$$

Theorem (R., Thurston)**.** No!

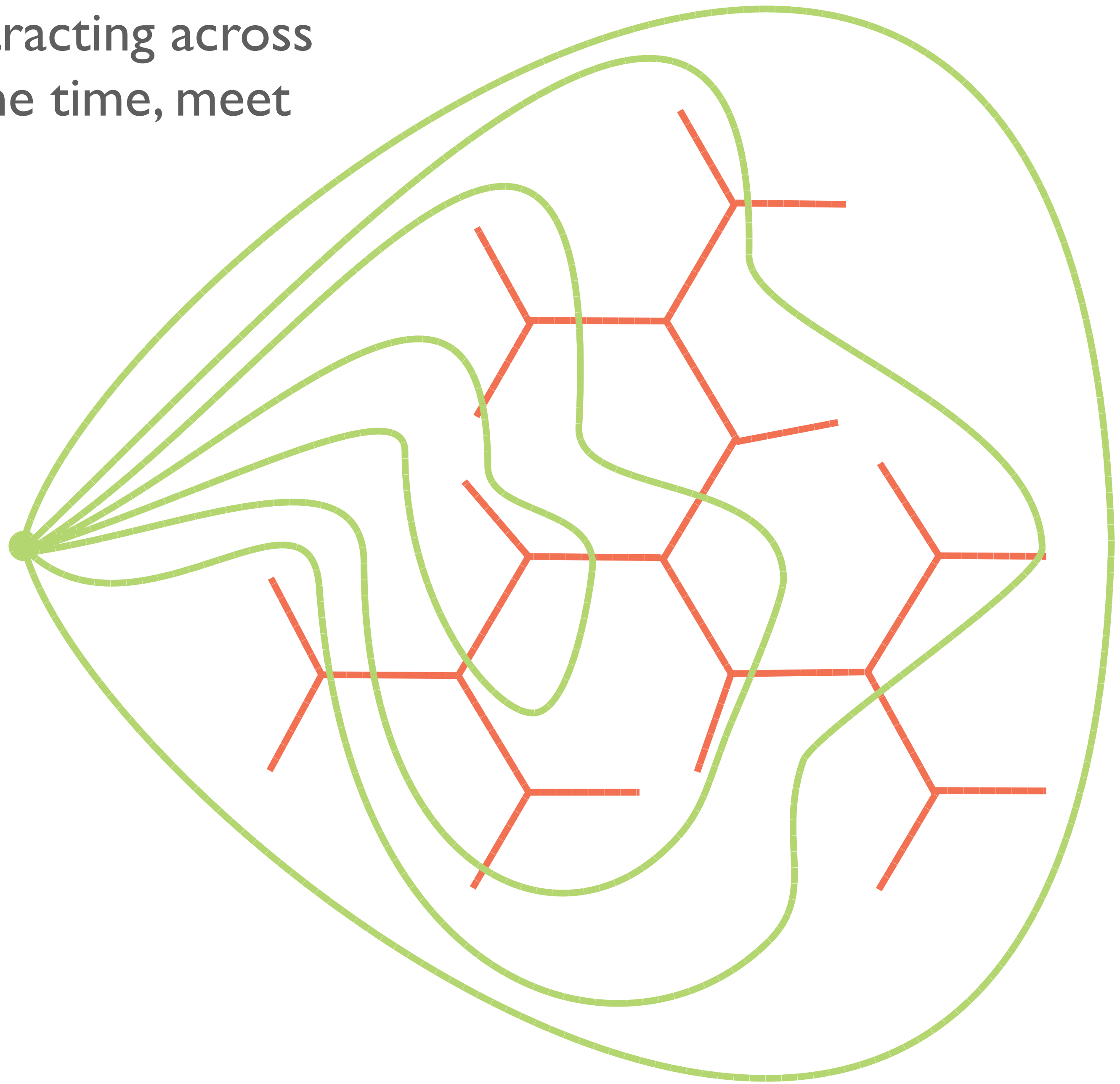
Proposition (Gersten, R.)**.** Suppose Δ is a combinatorial 2-disc in which every 2-cell has degree at most λ. If T is a spanning tree in $G = \Delta^{(1)}$ then

$$\mathrm{FL}(\Delta) \ \leq \ \mathrm{Diam}(T) + 2\lambda \mathrm{Diam}(T^*) + \ell(\partial\Delta).$$

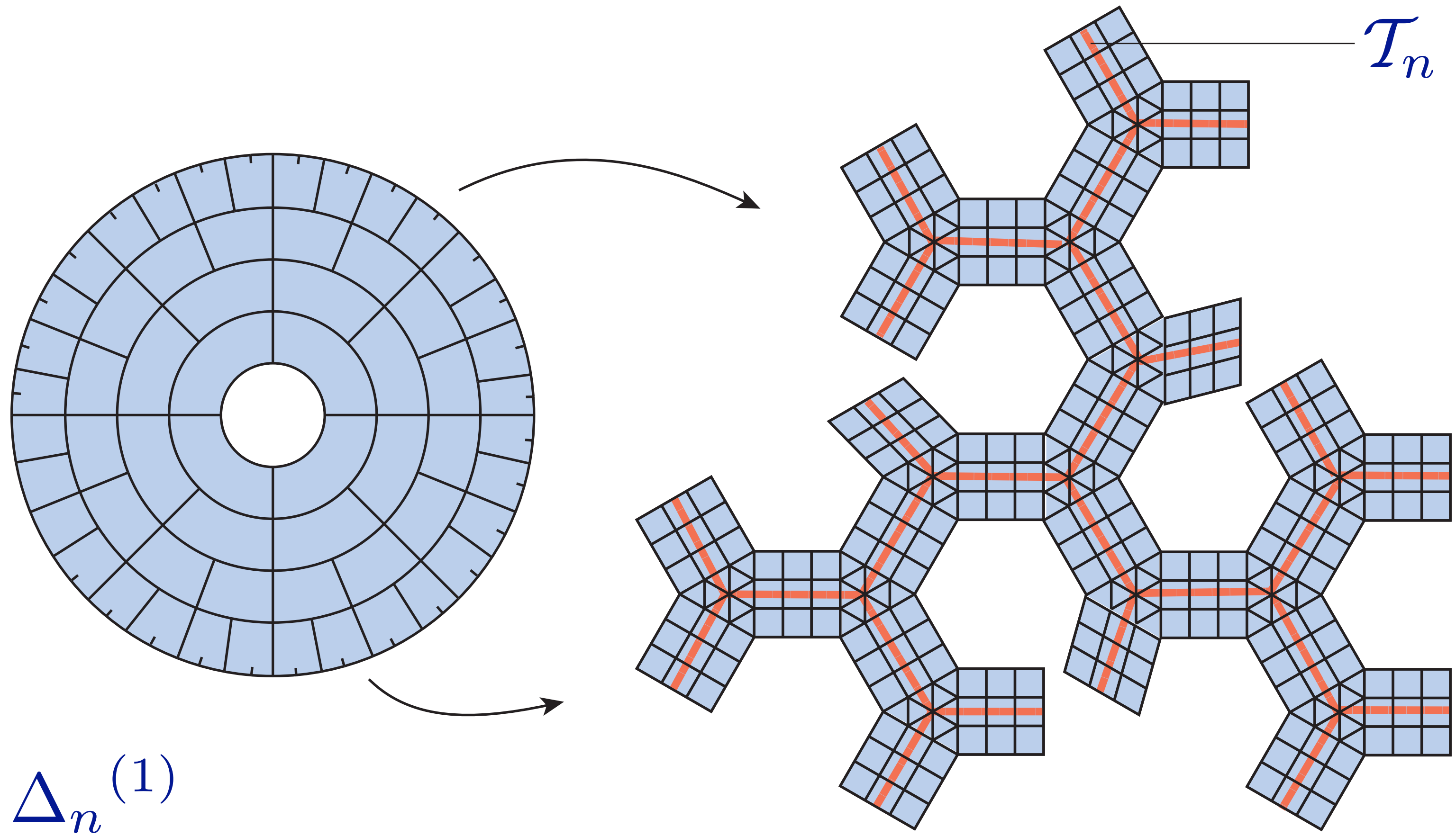


$\mathcal{T}_1$
$\mathcal{T}_2$
$\mathcal{T}_3$

Any loop contracting across $\mathcal{T}_n$ will, at some time, meet $n+1$ edges.



Δ_n



$G_n = {\Delta_n}^{(1)}$

$\mathrm{Diam}(G_n) \simeq \mathrm{Diam}({G_n}^*) \simeq n$

(cf. Frankel–Katz)

$\mathrm{FL}(\Delta_n) \succeq n^2$

So for all spanning trees T in G_n,

$$\mathrm{Diam}(T) + 10\mathrm{Diam}(T^*) + 4 \succeq n^2.$$

Application. Finding a finitely presented group for which

$$\mathrm{FL} \not\simeq \mathrm{IDiam} \quad \text{and} \quad \mathrm{FL} \not\simeq \mathrm{GL}.$$

Candidate. The group generated by

$$\ldots$$

subject to

$$\ldots$$

may have $\mathrm{GL}(n) \simeq \mathrm{IDiam}(n) \preceq n^3$ and $\mathrm{FL}(n) \succeq n^4$.