Contact 3-Manifolds and Geometric Topology

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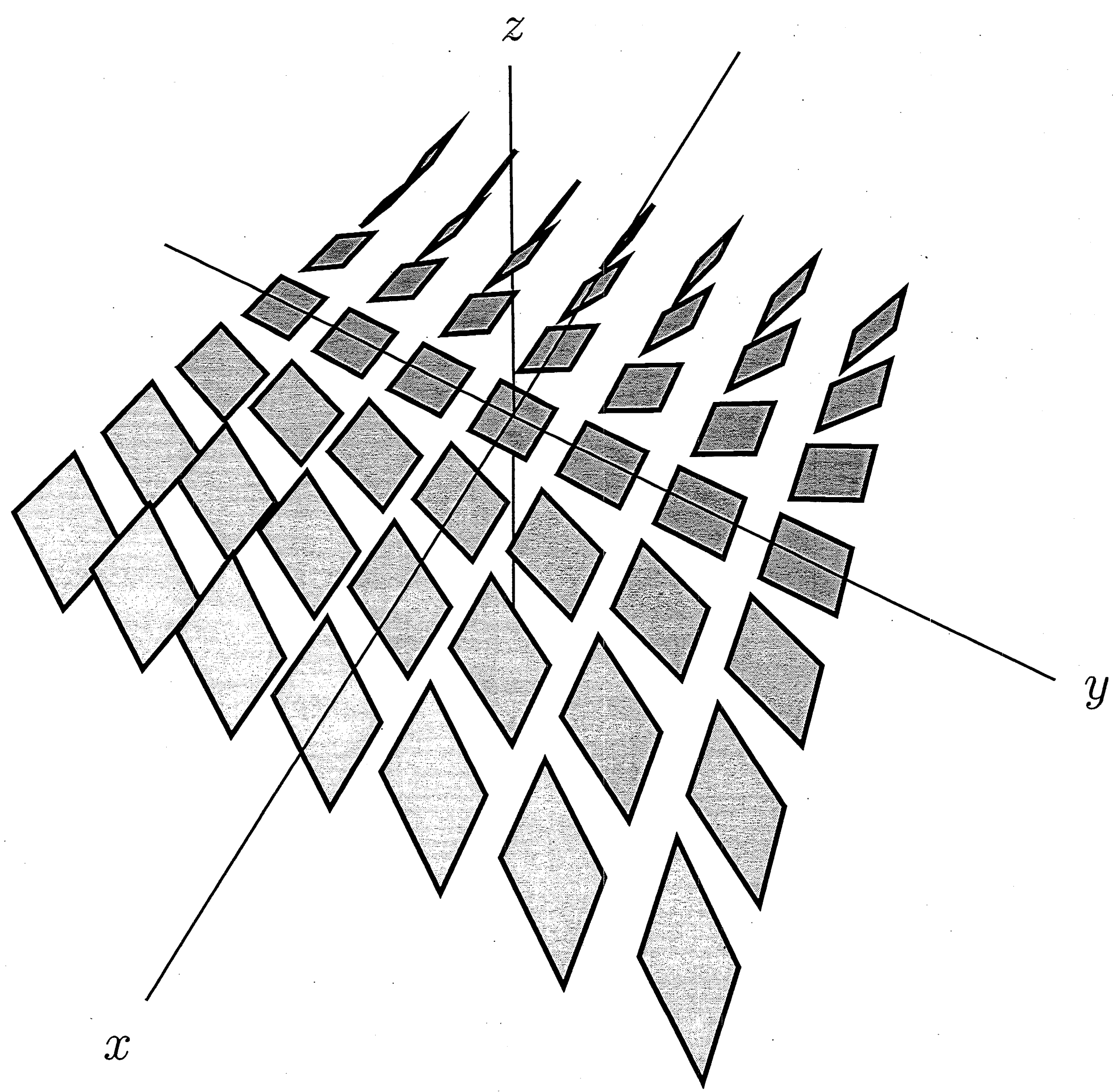
contact structure on M^3: $\xi = \ker \alpha$ with $\alpha \wedge d\alpha \neq 0$

Darboux theorem: Locally $\alpha = dz + x\,dy$



The contact structure $\xi = \ker(dz + x\,dy)$.

Theorem of Martinet: Every closed, orientable M^3 admits a contact structure.

Proofs:

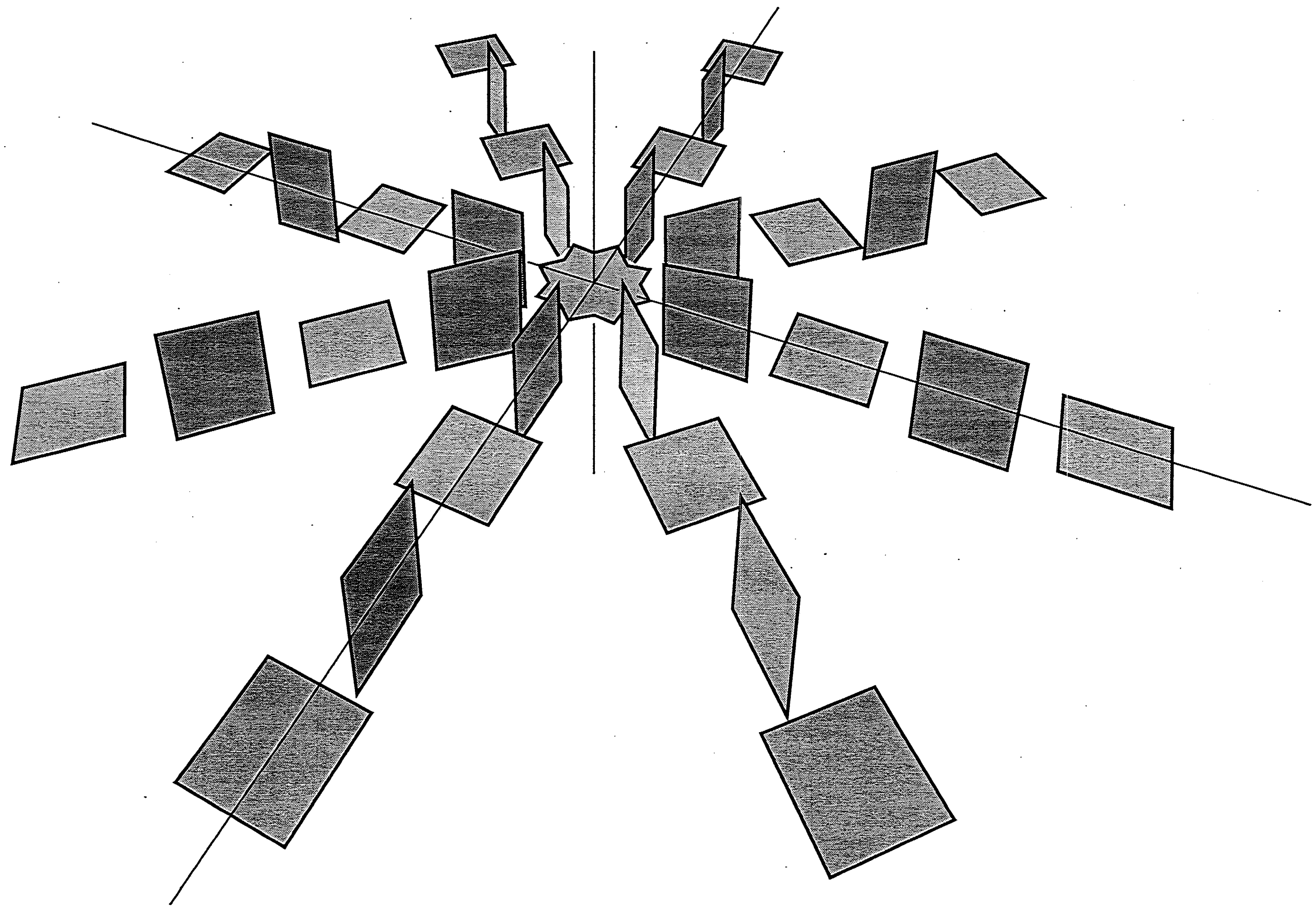
- surgery along transverse knots (Martinet, 1971)
- open books (Thurston–Winkelnkemper, 1975)
- branched covers (Gonzalo, 1987)
- surgery along Legendrian knots (Ding–G–Stipsicz, 2004)

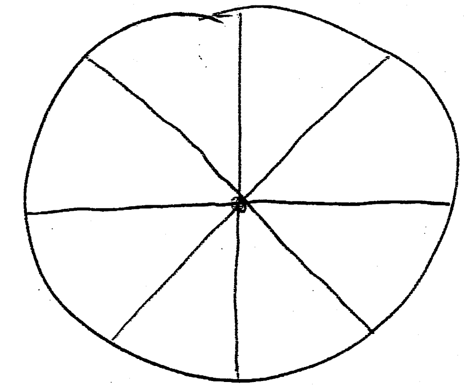
ξ is **overtwisted** if $\exists\, \Delta \hookrightarrow (M, \xi)$ with $T_p\Delta = \xi_p$ for all $p \in \partial\Delta$.

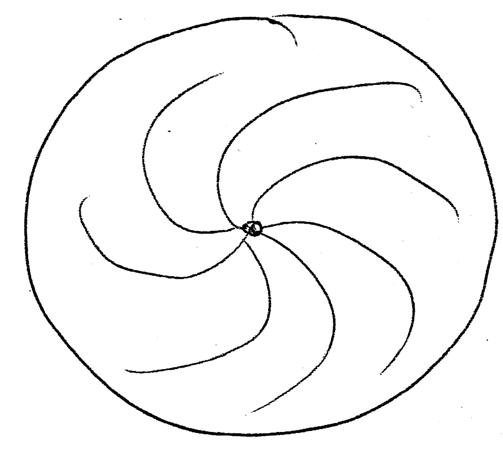
ξ is **tight** if it is not overtwisted.

Examples of **tight** contact structures:

- $\xi_{\text{st}} = \ker(dz + x\,dy)$ on $\mathbf{R}^3$
- $\xi_{\text{st}} = \ker(x\,dy - y\,dx + z\,dt - t\,dz)$ on S^3
- $\xi_n = \cos(n\theta)\,dx - \sin(n\theta)\,dy$ on T^3, $n \in \mathbf{N}$
- $z\,d\theta + x\,dy - y\,dx$ on $S^1 \times S^2$







The overtwisted contact structure

$\xi = \ker(\cos r\, dz + r \sin r\, d\varphi)$.

Two classical theorems of Eliashberg

- (1989) Classification of overtwisted contact structures is homotopical.

- (1992) Standard contact structure on S^3 is unique tight one up to isotopy.
  Application: Proof of **Cerf's Theorem** $\Gamma_4 = 0$

Prime decomposition theorem for tight contact 3-manifolds

Every closed tight contact 3-manifold can be written as a connected sum of prime tight contact 3-manifolds (Colin, 1997); the summands are unique up to order and contactomorphism (Ding–G, 2008).

Remark: The uniqueness statement fails for overtwisted contact 3-manifolds.

Surgery presentation theorem for contact 3-manifolds

Every closed contact 3-manifold can be obtained from (S^3, ξ_{st}) by **contact** (± 1)-**surgery** along a Legendrian link (Ding–G, 2004).

Symplectic fillings

- A compact symplectic manifold (W^4, ω) is a **weak filling** of (M^3, ξ) if $\partial W = M$ as oriented manifolds and $\omega|_\xi > 0$.

- A compact symplectic manifold (W^4, ω) is a **strong filling** of (M^3, ξ) if $\partial W = M$ and $\xi = \ker(i_X\omega)$ with X a Liouville vector field for ω, defined near ∂W, pointing outwards.

Liouville vector field: $L_X\omega \equiv d(i_X\omega) = \omega$

$(i_X\omega) \wedge d(i_X\omega) = (i_X\omega) \wedge \omega = (i_X\omega^2)/2 \neq 0$ on M transverse to X

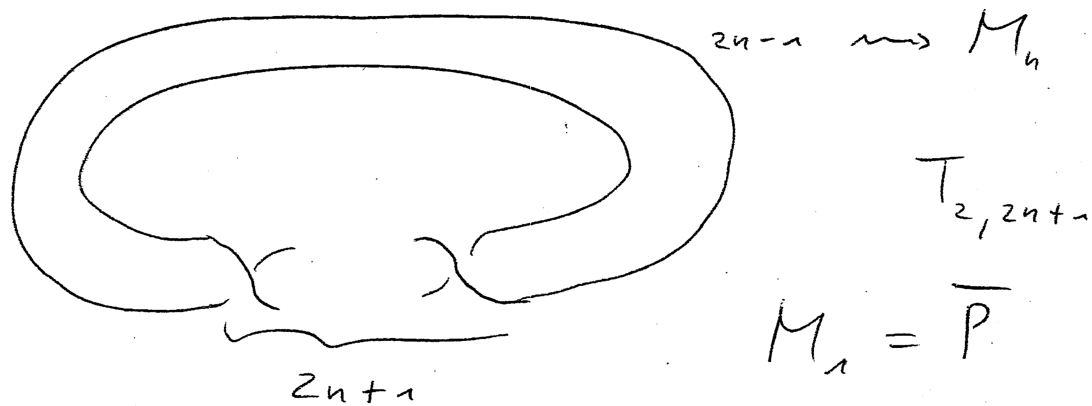
e.g. $W = D^4$, $\omega = dx \wedge dy + dz \wedge dt$, $X = r\partial_r$ defines strong filling of (S^3, ξ_{st}).

	clear		Eliashberg, Gromov, 1985	
	$\Rightarrow$		$\Rightarrow$	
strong filling		weak filling		tight
	$\not\Leftarrow$		$\not\Leftarrow$	
	Eliashberg, 1996 Ding–G, 2001		Etnyre–Honda 2002 Lisca–Stipsicz, Ghiggini, 2004/5	

Tight contact structures on Seifert fibred 3-manifolds

Every Seifert fibred 3-manifold admits a (positive) tight contact structure for at least one choice of orientation (Gompf, 1998).

$P\#\overline{P}$ does not admit any tight contact structure (Etnyre–Honda, 2001).

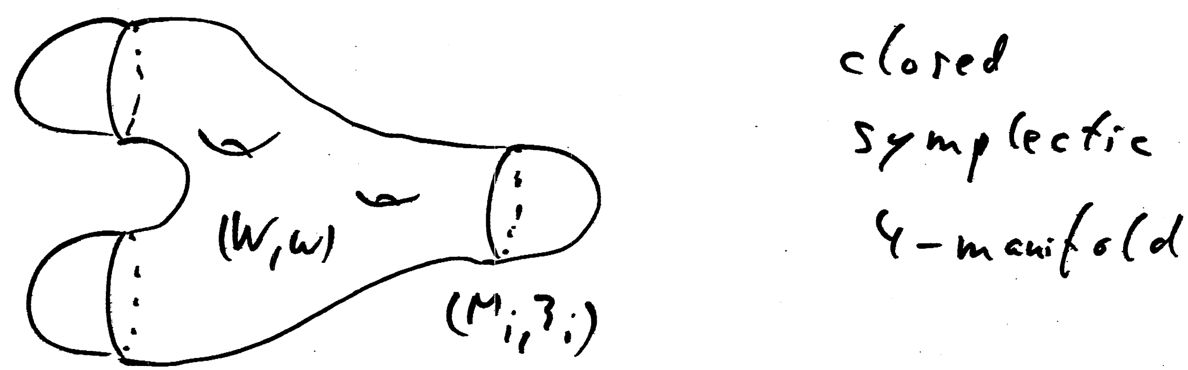


The only oriented Seifert 3-manifolds that do not admit a *positive* tight contact structure are the M_n (Lisca–Stipsicz, 2007).

Concave fillings and symplectic caps

- All contact 3-manifolds admit a strong concave filling (Gay, 2002).

- Weak fillings can be capped off (Eliashberg, Etnyre, 2004; Özbağcı–Stipsicz (G), 2006).



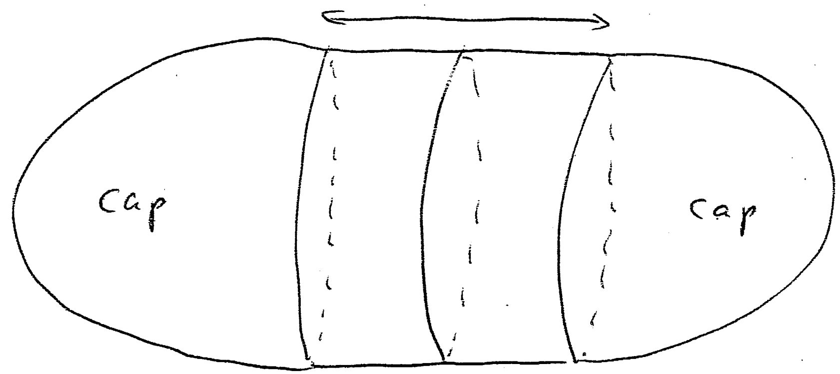
Symplectic capping-off enters the proofs of these results:

- Nontrivial knots have property P (Kronheimer–Mrowka, 2004).

- The unknot, trefoil and figure eight knot are determined by surgery (Kronheimer–Mrowka···Ozsváth–Szabó, 2007 *et seq.*)

- Heegaard Floer invariants detect minimal genus of surfaces representing $\beta \in H_2(M^3)$ (Thurston norm) and minimal Seifert genus of knots (Ozsváth–Szabó, 2004).

General strategy of proof



$(M^3 \times [-1, 1], \omega)$ weakly filling $M^3 \times \{\pm 1\}$

(Gabai, Eliashberg – Thurston)

M^3 e.g. from purported counterexample to Property P

Key tools in contact topological part

- Convex surface theory (Giroux, 1991).
- Open book decompositions (Giroux, 2002).

Convex surfaces

$\Sigma \subset (M^3, \xi)$ is **convex** if there is a contact vector field Y defined near and transverse to Σ. Then

$$\xi = \ker(\beta + u\,dt)$$

near Σ. Contact condition: $u\,d\beta + \beta \wedge du > 0$ $(*)$.

The **dividing set** of a convex surface Σ is

$$\Gamma := \{p \in \Sigma\colon Y(p) \in \xi_p\} = \{p \in \Sigma\colon u(p) = 0\}.$$

$(*) \Rightarrow du \neq 0$ along Γ, so this is a 1-dim. submanifold.

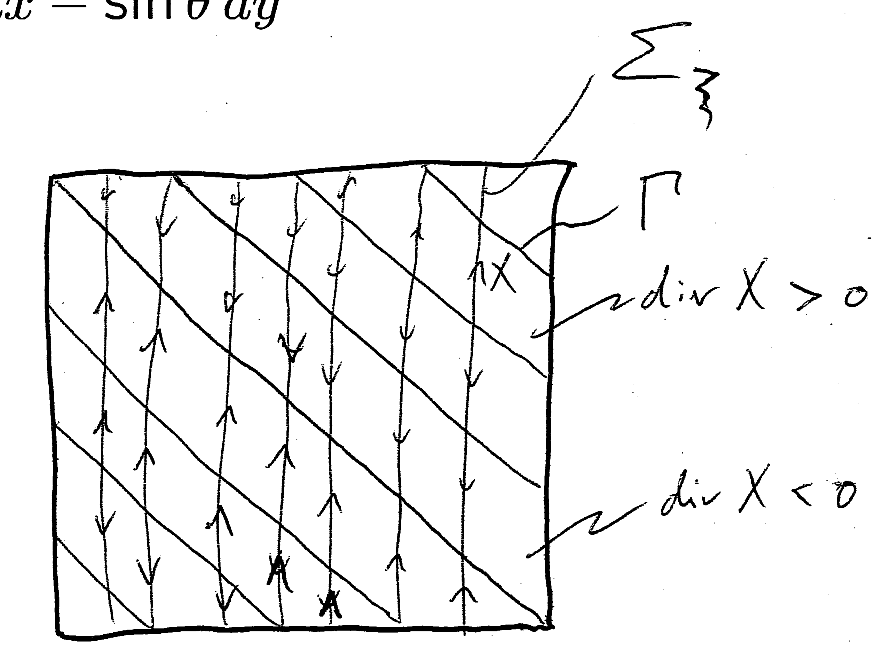
Γ essentially determines the germ of ξ near Σ.

Example of a convex surface

$\Sigma := S^1 \times \partial D^2 \subset S^1 \times \mathbf{R}^2, \quad \alpha := \cos\theta\, dx - \sin\theta\, dy$

$Y := x\,\partial_x + y\,\partial_y \Rightarrow L_Y\alpha = \alpha$

$\Gamma = \{x \cos\theta - y \sin\theta = 0\}$



Proof of uniqueness of tight structure on S^3 (Tomography)