Equivariant cohomology in Symplectic Geometry

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Cornell Unviersity
Topology Festival

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Hamiltonian T-spaces

Let M be a compact symplectic manifold, with symplectic form ω.
Suppose that T acts on M preserving ω.

Definition

For every $\xi \in \mathfrak{t}$, let X_ξ be the vector field on M generated by the ξ action. Suppose that the one-form $\omega(X_\xi, \cdot)$ is *exact*, i.e. there exists a function Φ^ξ satisfying

$$\omega(X_\xi, \cdot) = -d\Phi^\xi.$$

We may put these functions together to form a single map $\Phi : M \to \mathfrak{t}^*$ by letting $\Phi(p)(\xi) := \Phi^\xi(p)$ for every $p \in M, \xi \in \mathfrak{t}$. We say that Φ is a **moment map**. The existence of such a map is the statement that M is a **Hamiltonian** T space.

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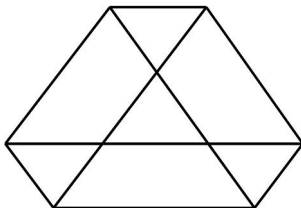
Convexity Theorem

Theorem (Guillemin-Sternberg, Atiyah, 1982)

Let M be a connected compact symplectic manifold with a Hamiltonian group action by an abelian Lie group T. Then the image of the moment map is a convex polytope. It is the convex hull of the image of the fixed point set.

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The Hero of the Story: S^2

Consider the 2-sphere $S^2 \subset \mathbb{R}^3$ with symplectic form at a point given by $\omega_p(u, v) = \langle p, u \times v \rangle$. Away from the poles, the form is given in polar coordinates by $d\theta \wedge dh$. S^1 acts on S^2 by rotating it around z-axis, generating the vector field $\frac{\partial}{\partial \theta}$.

There are two fixed points of points of the action, N and S. The moment map is the height function $h : S^2 \longrightarrow \mathbb{R}$ taking each point on S^2 to its z-value. Note that the image is a line segment, the convex hull of two points in $\mathbb{R}$.
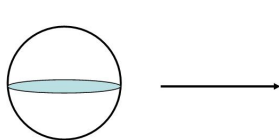
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Schur Horn theorem

The Schur-Horn Theorem (1923-1954). Let $\mathcal{O}_\lambda$ be the space of Hermitian matrices with spectrum $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let Φ take a matrix to its diagonal entries. Then the image of Φ is a convex polytope whose vertices are the n! permutations of λ.

In particular, if $p = (p_1, \ldots, p_n)$ is in the convex hull of these permutations, then there exists a matrix with spectrum λ and diagonal entries p.

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Examples of Hamiltonian T-spaces

- Even dimensional vector spaces

- Complex Projective space $\mathbb{C}P^n$.

- Flag Varieties, Grassmannians

- Coadjoint orbits of complex semi-simple Lie groups

- (Symplectic) toric varieties

If M is a Hamiltonian T-space, then $\dim M \geq 2 \dim T$. When $\dim M = 2 \dim T$, then we say that M is a **symplectic toric variety.**

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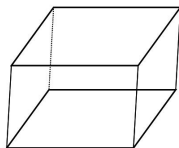# Delzant Polytopes and Toric Varieties

A Delzant polytope is a polytope in $\mathbb{R}^n$ such that

- there are n edges out of each vertex
- all edges point in rational directions
- the primitive vectors along the edges form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.
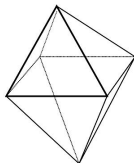
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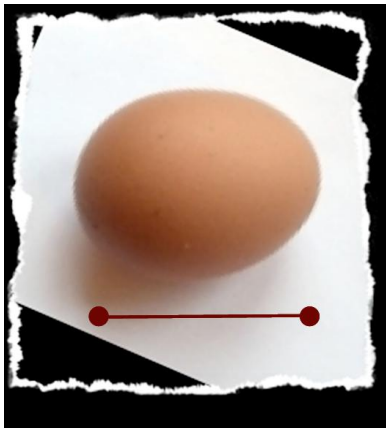
Orbifold toric varieties

Theorem. (Lerman-Tolman) Symplectic toric orbifolds are classified up to equivariant symplectomorphism by simple, rational polytopes with a positive integer associated to each facet. The bijection is via the moment map.
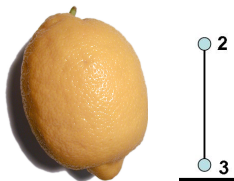
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An Open Question.

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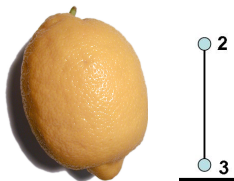

An Open Question.

Given a polytope in $\mathbb{R}^n$, perhaps decorated with additional information, is there a Hamiltonian T-space for which this is the image of a moment map?

GKM spaces

Suppose the T action on M has the following properties:

- T acts in a Hamiltonian fashion
- T acts with isolated fixed points M^T
- For every codimension one subtorus K in T, the connected components of M^K are at most two dimensional.

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Equivariant cohomology: one definition.

The Borel construction

Let BG be the classifying space of G, and EG the total space of the universal bundle over $BG = EG/G$. Note that G acts freely on EG. Thus G acts freely on MxEG via the anti-diagonal action. Consider the quotient $M_G := M \times_G EG$. The equivariant cohomology of M is by definition the singular cohomology of M_G i.e.

$$H_G^*(M) = H^*(M_G).$$

When $G = T$ is a torus, then $BT \simeq (\mathbb{C}P^\infty)^d$, $d = \dim T$.

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Properties of abelian equivariant cohomology
$H_T(M) := H^*(M \times_T ET)$

- The squish map $M \rightarrow pt$ induces a module structure $H_T^*(pt) \rightarrow H_T^*(M)$ on M. When M is a Hamiltonian T space, $H_T^*(M)$ is a free module over $H_T^*(pt)$.

- For T d-dimensional torus acting trivially on a point p,

$$H_T^*(p) = H^*(ET/T) = H^*(BT) = \mathbb{C}[u_1, \ldots, u_d].$$

- If G acts freely on a manifold Y, then $H_G^*(Y) = H^*(Y/G)$.

- When both a compact connected Lie group G and a maximal abelian subgroup $T \subset G$ act continuously on a topological space M,

$$H_G^*(M; \mathbb{Q}) = H_T^*(M; \mathbb{Q})^W$$

where W is the Weyl group.

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Let T act on M in a Hamiltonian fashion.

Let M^T denote the fixed point set. Then the inclusion $M^T \hookrightarrow M$ induces an **injection**:

$$H_T^*(M) \to H_T^*(M^T).$$

Compare to ordinary cohomology, where this would be absurd! Note that when M^T is finite, $H_T^*(M)$ injects into the direct sum of polynomial rings, one for each fixed point.

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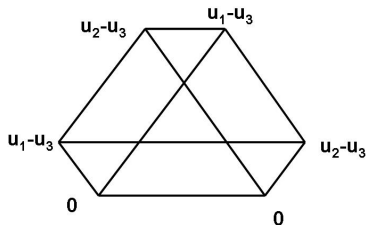
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Equivariant Cohomology for GKM spaces

Suppose that M is a GKM space. For every $K \subset T$ a codimension-1 torus, and every connected component M^K, associate the weight η_{pq} of the T action on $T_p(M^K)$. Note that $\eta_{pq} \in \mathfrak{t}^*$ and so naturally lives in $H_T^*(pt) = S(\mathfrak{t}^*)$. Then the image of the map

$$H_T^*(M) \longrightarrow H_T^*(M^T) = \bigoplus_{r \in M^T} H_T^*(r) = \bigoplus_{r \in M^T} \mathbb{C}[u_1, \ldots, u_d]$$

is given by those $\beta \in \bigoplus_{r \in M^T} \mathbb{C}[u_1, \ldots, u_d]$ such that

$$\beta(p) = \beta(q) \bmod \eta_{pq}.$$

This is not true over $\mathbb{Z}$.

The S^1-equivariant cohomology of S^2.

Therefore, the cohomology of S^1 in $\mathbb{C}[u] \oplus \mathbb{C}[u]$ is generated as a module over $\mathbb{C}[u]$ by $(u, 0)$, $(0, u)$ and $(1, 1)$.

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Equivariant cohomology: toric varieties

The cohomology of toric varieties is completely determined by the moment polytope. There is a generator in degree two for each facet of the polytope. Each such class is the Euler class of the normal bundle to the corresponding divisor. The relations come from the intersection properties of the facets on the polytope. This is how the (equivariant) Stanley-Reisner ring is formed.

The Sphere Again

From this point of view, we have the generators of $H^*_{S^1}(S^2)$ are the class 1, Euler class associated to the North pole, and the Euler class associated to the South pole. And the product of these two latter two classes is 0, since the facets do not intersect.

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Fixed Point Theorems

Atiyah-Bott-Berline-Vergne

Let $\alpha \in H_T^*(M)$. Then

$$\int_M \alpha = \sum_F \int_F \frac{\alpha(F)}{e_T(\nu(F))},$$

where the sum is over all components of the fixed point set. If the fixed points are all isolated, the formula is just a sum of rational functions. The amazing thing is that this big sum of rational functions is actually polynomial.

Another sum of rational functions is polynomial

Theorem (G-Tolman)

Let a torus T act on a compact symplectic manifold (M, ω) with isolated fixed points and moment map $\Phi \colon M \to \mathfrak{t}^*$. Fix a generic $\xi \in \mathfrak{t}$. Assume that there exists a canonical class $\alpha_p \in H_T^{2\lambda(p)}(M; \mathbb{Q})$ for all $p \in M^T$. Define an oriented graph with vertex set M^T and edge set

$$E = \left\{ (r, r') \in M^T \times M^T \,\middle|\, \lambda(r') - \lambda(r) = 1 \text{ and } \alpha_r(r') \neq 0 \right\}.$$

Let Σ_p^q be the set of paths from p to q in (V, E). Define $a(r, r') = \frac{\alpha_r(r')}{\Lambda_{r'}^-}$ for each edge $(r, r') \in E$. Then for all $p, q \in M^T$,

$$\alpha_p(q) = \Lambda_q^- \sum_{\mathbf{r} \in \Sigma_p^q} \prod_{i=1}^k \frac{\Phi(r_i) - \Phi(r_{i-1})}{\Phi(q) - \Phi(r_{i-1})} a(r_{i-1}, r_i).$$

Similar formula was found by Guillemin-Zara (but with more paths).

S. Billey

When M is a G/B, for G a complex reductive Lie group and B a Borel subgroup. Let $\mathcal{S}_p$ be a canonical class (Schubert class). They are indexed by fixed points, which are indexed by elements in the Weyl group. For any q, fix a reduced word $\mathbf{Q} = b_1 \cdots b_r$ expression for q, and

$$\mathcal{S}_p(q) = \sum_{b_{i_1} b_{i_2} \cdots b_{i_k} \in R(p)} \prod_{j=1}^{k} r_{b_1} r_{b_2} \cdots r_{b_{i_{j-1}}} \alpha_{b_{i_j}}$$

where $R(p)$ are the set of all reduced words for p and α_i is the root corresponding to the simple reflection r_i.

Equivalence between fixed point restrictions and structure constants

Given a preferred (module-)basis for the equivariant cohomology, the restriction to the fixed points gives you all the information you need to multiply in this basis.

Suppose $\{S_i\}$ is a $H_T^*(pt)$-basis for $H_T^*(M)$. Then structure constants $c_{ij}^k \in H_T^*(pt)$ are defined by

$$S_i S_j = \sum_k c_{ij}^k S_k.$$

Inductively, can find a formula for c_{ij}^k when you know how S_i restricts to each $p \in M^T$ for all i.

But it's not a "positive relationship".

This tells us the ordinary cohomology structure constants when M is Hamiltonian, since $H_T^*(M) \to H^*(M)$ is surjective.

Quotients by group actions can be messy

We would like to quotient our manifold by its symmetries, but this is topologically messy.

Consider the $\mathbb{C}^*$ action on $\mathbb{C}$ given by multiplication by complex numbers. The quotient space has two orbits, 0 and everything else. Its not Hausdorff. The idea of Geometric Invariant Theory (GIT) is to rip out the bad point (0 in this case) and then the action is just $\mathbb{C}^*$ on $\mathbb{C}^*$. The quotient is a point.

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Symplectic reduction

Definition

Let M be a compact symplectic manifold with a Hamiltonian T action and moment map Φ. The **symplectic reduction** of M at a value μ is the quotient

$$M /\!/ T(\mu) := \Phi^{-1}(\mu)/T.$$

The symplectic quotient inherits a symplectic form ω_μ from M.

A new description of S^2.

Let S^1 act on $\mathbb{C}^2$ by $\theta \cdot (z_1, z_2) = (e^{2\pi i\theta} z_1, e^{2\pi i\theta} z_2)$. This action is Hamiltonian with moment map $\Phi : \mathbb{C}^2 \to \mathbb{R}$ given by

$$\Phi : (z_1, z_2) \mapsto \frac{1}{2}(||z_1||^2 + ||z_2||^2).$$

Then $\Phi^{-1}(2) \subset \mathbb{C}^2$ is the three sphere with radius 1. Note that S^1 also acts on this 3-sphere, and the quotient forms S^2 (via the Hopf fibration).

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Symplectic reduction, cont.

Examples include toric varieties, Grassmannians and $\mathbb{C}P^n$, weight varieties, polygon spaces, the moduli space of flat connections (mod gauge equivalence), moduli space of points on $\mathbb{C}P^n$.

Equivariant cohomology and Kirwan surjectivity

Let T act on M in a Hamiltonian fashion with proper moment map $\Phi : M \to \mathfrak{t}^*$.

Theorem (Kirwan).

The inclusion $\Phi^{-1}(\mu) \hookrightarrow M$ induces a surjection:

$$\kappa_\mu : H_T^*(M) \to H_T^*(\Phi^{-1}(\mu)) \cong H^*(M /\!/ T).$$

Also fails for ordinary cohomology in simple examples.

Surjectivity (the Kirwan map)

The kernel of the surjective map

$$\kappa_\mu : H_T^*(M) \to H_T^*(\Phi^{-1}(\mu)) \cong H^*(M/\!/T)$$

can be computed! Describe the elements in $ker(\kappa_\mu)$ by describing their image in $H_T^*(M^T)$.

Tolman-Weitsman

The kernel of κ_μ is generated by those classes $\alpha \in H_T^*(M)$ such that α restricts to 0 on all fixed points to one side of an affine hyperplane that goes through μ.

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Jeffrey-Kirwan Localization and consequences

JK Localization for isolated fixed points

Let $\beta \in H_T^*(M)$, and let $\kappa_\mu : H_T^*(M) \to H^*(M//T)$. Then

$$\int_{M//T} \kappa_\mu(\beta)e^{\omega_\mu} = c \cdot Res^\Lambda \sum_{p \in M^T} e^{i(\Phi(p)-\mu)} \frac{\iota_p^*(\beta(X)e^\omega)}{e_p}[dX]$$

where c is a nonzero constant, Res^Λ is a multi-dimensional residue, $X \in \mathfrak{t} \otimes \mathbb{C}$ is a variable, and e_p is the equivariant Euler class of the normal bundle to p.

The original purpose of this formula was to find intersection pairings on the moduli space of holomorphic vector bundles on a Riemann surface.

Another consequence of JK Localization

G-Holm-Jeffrey

The ideal $I_\mu := \ker \kappa_\mu$ is an invariant of the chambers of the moment map. In other words, μ and ν are in distinct chambers of the image of Φ if and only if $I_\mu = I_\nu$ in $H_T^*(M)$.
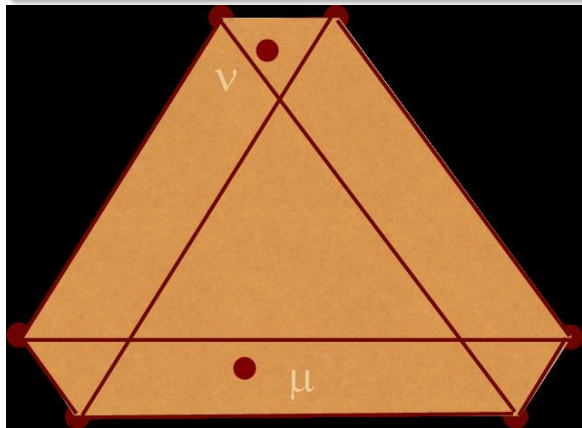
S^2 as a symplectic reduction

Let $S^1 \times S^1$ act on $\mathbb{C}^2$ by $(\theta_1, \theta_2) \cdot (z_1, z_2) = (e^{2\pi i \theta_1} z_1, e^{2\pi i \theta_2} z_2)$. We saw earlier that S^2 is formed as a symplectic quotient of $\mathbb{C}^2$ by the diagonal $S^1 \subset S^1 \times S^1$. We obtain an equivariant Kirwan map

$$H_T^*(\mathbb{C}^2) \to H_{S^1}^*(S^2)$$

where (careful!) this S^1 is the residual S^1 acting on S^2. The kernel of the map can be computed as in the non-equivariant Kirwan map.

S^1-equivariant cohomology of S^2 revisited

The (equivariant) TW theorem tells us that we should take the cohomology of $\mathbb{C}^2$ and quotient by the ideal generated by classes which are 0 to one side of a hyperplane through 1. This consists of (any multiple of) the $T = S^1 \times S^1$-equivariant Euler class to $\{0\} \in \mathbb{C}^2$. Thus

$$H_{S^1}^*(S^2) = H_T^*(\mathbb{C}^2) / \langle e_T(\{0\}) \rangle = \mathbb{C}[u_1, u_2] / \langle u_1 u_2 \rangle.$$

When it comes to orbifolds...

Symplectic quotients at regular values are always either manifolds or orbifolds.

The Lemon

Let S^1 act on $\mathbb{C}^2$ by $\theta \cdot (z_1, z_2) = (e^{2 \cdot 2\pi i\theta} z_1, e^{3 \cdot 2\pi i\theta} z_2)$. The moment map $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{R}$ is given by

$$\Phi : (z_1, z_2) \mapsto \frac{1}{2}(2||z_1||^2 + 3||z_2||^2).$$

The preimage of a regular value (say, 1) is a 3-dimensional ellipsoid. For a generic point $(z_1, z_2) \in \Phi^{-1}(1)$, the group has no isotropy (i.e. the stabilizer of (z_1, z_2) is generically just $1 \in S^1$). However, at a point $(z_1, 0)$, there is $\mathbb{Z}_2$ isotropy, and at a point $(0, z_2)$ there is $\mathbb{Z}_3$ isotropy. In the quotient space, these descend to two orbi-points on a sphere-like object.

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Cohomology fails the lemon

Ordinary cohomology does not see this orbifold structure, and won't distinguish between the lemon and the egg. So Chen and Ruan introduced a new theory of cohomology (that is not a "cohomology theory"), now called Chen-Ruan cohomology. In particular,

$$H^*_{CR}(S^2_{(2,3)}) \neq H^*_{CR}(S^2).$$

Is there an algebraic invariant that would be subtle enough to detect differences in orbifolds?

Chen-Ruan cohomology

Let T act locally freely on a compact stably complex manifold Z, and let $X = Z/T$ be the quotient orbifold. Let

$$\widetilde{Z} = \{(z,g) \mid z \in Z, g \in T, g \cdot z = z\} \subset Z \times T.$$

Note that $Z \times 1 \subset \widetilde{Z}$.

Since T is abelian, T acts on $\widetilde{Z}$, and we denote the quotient by $\widetilde{X}$. Then

$$H^*_{CR}(X) := H^*(\widetilde{X}) \quad \text{as a group.}$$

The product on Chen-Ruan cohomology

The product is difficult to define because it involves an *obstruction bundle*. It is nontrivial to show that the product introduced by Chen-Ruan is associative. Chen-Ruan cohomology is not a cohomology theory. For example, an equivariant inclusion of orbifolds does not necessarily induce a ring map in CR-cohomology.

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Inertial cohomology: a definition without the ring structure

The case of a locally free action suggests a beginning:
$H^*_{CR}(X) := H^*(\widetilde{X}) = H^*_T(\widetilde{Z})$.

Definition of inertial cohomology (G-Holm-Knutson, Chen-Hu

Let Y be a stably almost complex manifold with a T action (that is not necessarily locally free). Define the inertial cohomology

$$NH^*_T(Y) := \bigoplus_{g \in T} H^*_T(Y^g).$$

This is also endowed with a twisted product, not the usual one from equivariant cohomology. Note that when T acts on Y locally freely, the right hand side is $H^*_T(\widetilde{Y})$.

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In the case that Y is a Hamiltonian T space, the funny ring structure on $NH^*_T(Y)$ is easy to describe by (injectively) sticking it into the direct sum of the fixed points (which we assume here to be finite):

$$NH^*_T(Y) := \bigoplus_{g \in T} H^*_T(Y^g) \hookrightarrow \bigoplus_{g \in T} \bigoplus_{p \in (Y^g)^T} H^*_T(p).$$

Inertial Cohomology: where to do computations

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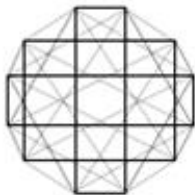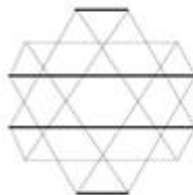

(a)   (b)

A product on inertial cohomology

Under this injection, the product is described by

$$(b_1 \star b_2)|_{F,g} := \sum_{\substack{(g_1, g_2): \\ g_1 g_2 = g}} (b_1)|_{F, g_1} (b_2)|_{F, g_2} \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(g_1) + a_\lambda^F(g_2) - a_\lambda^F(g_1 g_2)}$$
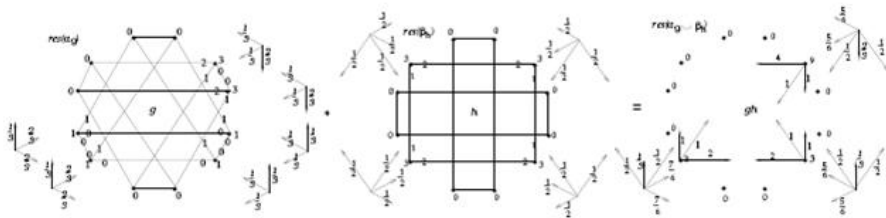
where $a_\lambda^F(g_1) \in [0, 1)$ is the real number such that g_1 acts on the I_λ-component of νF with eigenvalue $e^{2\pi i a_\lambda^F(g_1)}$.

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Kirwan surjectivity in orbifold cohomology

Theorem (G-Holm-Knutson).

Let Y be a Hamiltonian T space, $\Phi : M \to \mathfrak{t}^*$ a moment map, and $\mu \in \mathfrak{t}$ a regular value of Φ. Then there is a surjection of rings

$$\kappa_{NH} : NH_T^*(Y) \longrightarrow H_{CR}^*(\Phi^{-1}(\mu)/T) = H_{CR}^*(M /\!/ T(\mu)).$$

The kernel of κ_{NH}.

The kernel of the map can be computed in exactly the same way that it was done by Tolman-Weitsman for equivariant cohomology! The combinatorics are messy but turn out to be 'easy'.

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More surjectivity for orbifold cohomology

Surjectivity for toric varieties

One consequence is the CR-cohomology for (most) orbifold toric varieties, sometimes over $\mathbb{Z}$. This was independently calculated by Borisov-Chen-Smith (over $\mathbb{Q}$). We showed that over $\mathbb{Z}$ the CR-cohomology of an orbifold and the usual cohomology of its crepant resolutions do not even agree as vector spaces.

Theorem (G-Harada). The hypertoric case

The hyperkähler analogue of κ_{NH} is also surjective, and the kernel can be computed using combinatorial methods. Results in the CR-cohomology of hypertoric varieties. Jiang-Tseng got similar results using different techniques.

Back to the Lemon

Recall S^1 acts on $\mathbb{C}^2$ with weight 2 on the first component, and weight 3 on the second. Then $\Phi : \mathbb{C}^2 \to \mathbb{R}$ is given by $(z_1, z_2) \mapsto 2||z_1||^2 + 3||z_2||^2$. The preimage of a regular value is an ellipsoid Z. Let $X = Z/S^1$ be the lemon. Then $NH_{S^1}(Z) = H_{CR}(X)$, and $NH_{S^1}(\mathbb{C}^2) \twoheadrightarrow NH_{S^1}(Z)$.
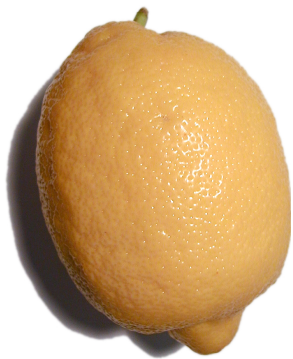
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The CR cohomology is
$\mathbb{Q}[u, \beta, \gamma, \delta]/\langle u\beta, u\gamma, u\delta, u^2 \rangle$.

Some open questions

- Is there a way to define inertial cohomology for Hamiltonian *G*-spaces, where *G* is not abelian? Does it surject onto the Chen-Ruan cohomology?
- Is there a good definition of the equivariant cohomology of an orbifold? If so, what are the combinatorial invariants we can use to compute it?
- What are all the K-theoretic analogs of these statements? What are the implications for representation theory?