Morse-Witten theory

$f : X \to \mathbb{R}$, choose metric on X

closed manifold ($\dim < \infty$) , oriented

- **critical points**

$$\mathrm{Crit}\, f = \{ x \in X \mid \nabla f(x) = 0 \}$$

$$\rightsquigarrow C_* = \bigoplus_{x \in \mathrm{Crit} f} \mathbb{Z}\, x$$

- **negative gradient flow lines**

$$\rightsquigarrow \partial : C_* \to C_*$$

(signed count)

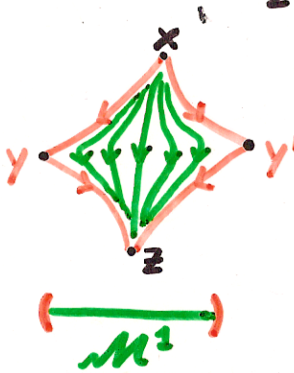
$$\partial x = \sum_{y \in \mathrm{Crit} f} \# \left\{ \begin{array}{l} \underline{\text{isolated}} \text{ flow lines} \\ \text{from } x \text{ to } y \end{array} \right\} y$$

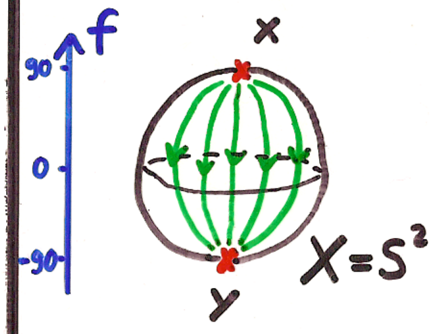
Thm.: $\partial \circ \partial = 0$

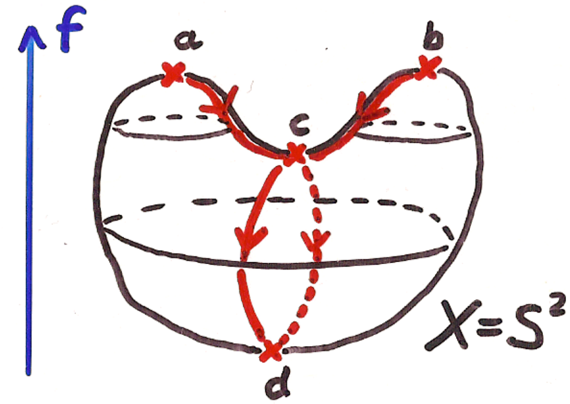
$$\Rightarrow HM_*(X;f) = {}^{\ker \partial}\!/\!_{\mathrm{im}\, \partial} \quad \text{exists}$$

Proof: $$\partial^2 x = \sum_{z \in \mathrm{Crit} f} \# \left\{ \begin{array}{l} \text{broken flow lines} \\ \text{from } x \text{ to } z \end{array} \right\} z$$

$$\| \qquad \# \{ \text{ends of } \mathcal{M}^1(x,z) \} \qquad \| \qquad \underline{\underline{0}}$$

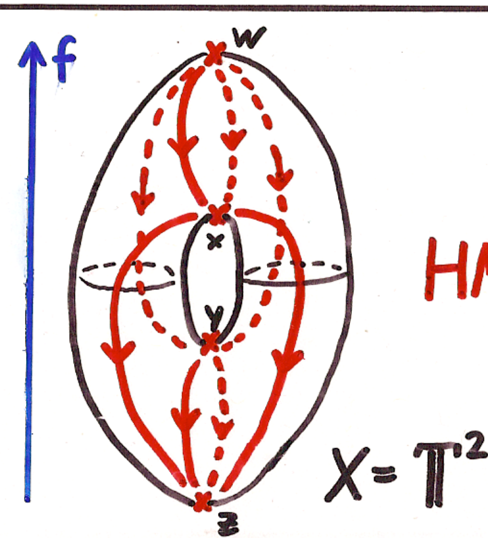






$\mathbb{Z}x \oplus \mathbb{Z}y$	$\mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}d$
$\partial x = 0$ $\partial y = 0$	$\partial a = c \quad \partial c = d - d = 0$ $\partial b = -c \quad \partial d = 0$
$\ker \partial = \mathbb{Z}x \oplus \mathbb{Z}y$ $\mathrm{im}\, \partial = 0$	$\ker \partial = \mathbb{Z}(a+b) \oplus \mathbb{Z}c \oplus \mathbb{Z}d$ $\mathrm{im}\, \partial = \mathbb{Z}c$

$$\underline{\mathbb{Z}x \oplus \mathbb{Z}y} \cong HM_*(S^2) \cong \underline{\mathbb{Z}(a+b) \oplus \mathbb{Z}d}$$



$C_* = \ldots\ldots$

$\partial = 0$

$$HM_*(\mathbb{T}^2) = \underline{\mathbb{Z}w \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z}$$

Floer theory ≙ "∞-dim. Morse theory"

[Andreas Floer 1980's]

(1) ~~periodic symplectic~~ Hamiltonian Floer homology

$\mathcal{X}^0_M = \{\gamma : S^1 \to M\}$ loop space of M symplectic manifold
contractible (with L^2-metric) "classical mechanics"

$f_H : \mathcal{X}^0_M \to \mathbb{R}$ action functional of H Hamiltonian
$H : S^1 \times M \to \mathbb{R}$ "energy"

$$\nabla f_H : \begin{array}{ccc} \mathcal{X} & \longrightarrow & \text{"}T\mathcal{X}\text{"} \\ \gamma & \longmapsto & J\dot{\gamma} - \nabla H(\gamma) \end{array} \quad ; \quad \begin{array}{l} J : TM \to TM \\ J^2 = -\mathbb{1} \end{array}$$

(ω-compatible almost complex structure)

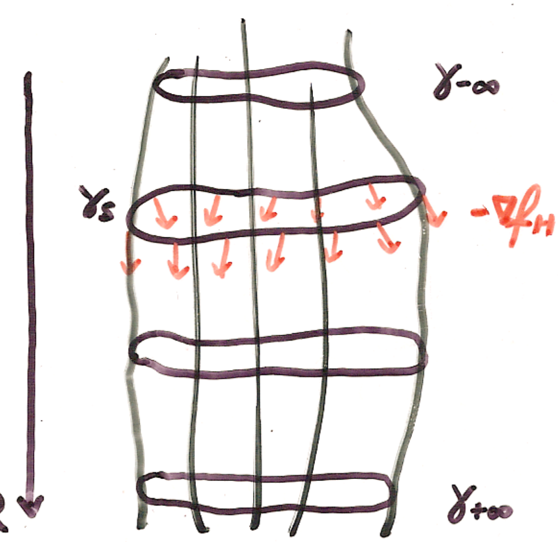
critical points:

periodic orbits γ of Hamiltonian : $\dot{\gamma} + J\nabla H(\gamma) = 0$

negative gradient flow lines:

$\gamma : \mathbb{R} \to \mathcal{X}^0_M = Map(S^1, M)$

$\frac{d}{ds}\gamma_s = -\nabla f_H(\gamma_s)$ ill posed ODE



Floer trajectories:

$u : \mathbb{R} \times S^1 \longrightarrow M$

$\frac{\partial}{\partial s} u + J \frac{\partial}{\partial t} u - \nabla H(u) = 0$

∇f_H is a differential operator, not a vector field

Thm (Floer, ...): $HM_*(\mathcal{X}^0_M, f_H) =: HF_*(M, H)$ exists

.... and is $\cong HM_*(M, H) \cong H_*(M)$.

⇒ **Arnold conjecture** : Hamiltonian H has $\geq \sum \operatorname{rk} H_i$ orbits

symplectic action functional

- **M exact** (e.g. $\mathbb{R}^{2n}, T^*X$) : $\qquad \omega = d\lambda$

$$f_H(\gamma) := -\int_{S^1} \gamma^*\lambda \;-\; \int_{S^1} H_t(\gamma(t))\,dt$$

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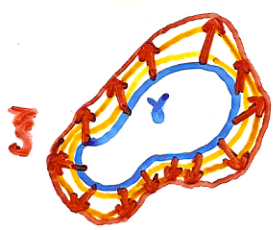
gradient: $\frac{d}{d\tau}\Big|_{\tau=0} f_H(\exp_\gamma(\tau\cdot\xi)) = \langle \xi, \boxed{\nabla f_H(\gamma)} \rangle_{L^2(S^1)} \quad \forall \xi \in \gamma^*TM$



$v(s,t) = \exp_{\gamma(t)}(s\cdot\xi(t))$

$$= \frac{d}{d\tau}\Big|_{\tau=0} \Big(\underbrace{-\int_{S^1} \exp_\gamma(\tau\cdot\xi)^*\lambda + \int_{S^1} \exp_\gamma(0\cdot\xi)^*\lambda}_{-\int_{[0,\tau]\times S^1} d(v^*\lambda) \;=\; -\int_{[0,\tau]\times S^1} v^*\omega} \Big) - \int_{S^1} dH_{t}(\gamma(t))(\xi(t))\,dt$$

$$= -\int_{S^1} \omega(\underbrace{\partial_s v}_{=\xi}, \underbrace{\partial_t v}_{=\dot\gamma})\Big|_{s=0} dt \;-\; \int_{S^1} \langle \xi(t), \nabla H_t(\gamma(t)) \rangle\, dt$$

$-J^2 = 1$

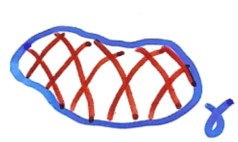
compatibility
$\omega(\cdot, J\cdot) = \langle \cdot, \cdot \rangle$

$$= \langle \xi, \boxed{J\dot\gamma - \nabla H(\gamma)} \rangle$$

- **general (compact) M** :

$$f_H(\gamma) := -\int_{D^2} v^*\omega \;-\; \int_{S^1} H_t(\gamma(t))\,dt \quad \in \mathbb{R}/_{\langle [\omega], \pi_2(M) \rangle}$$

$v : D^2 \to M$
$v|_{\partial D^2} = \gamma$



Transversality & compactness for Ham. Floer homology

- f_H is "<u>Morse</u>" (Hessian $\nabla^2 f_H$ at crit. pts. is nondegenerate) for "generic" $H : S^1 \times M \to \mathbb{R}$.

- $(f_H, L^2\text{-metric w.r.t. } \omega, J)$ is ""<u>Morse-Smale</u>""

[i.e. moduli spaces of trajectories
$\mathcal{M}^k(x^-, x^+) = \{u : \mathbb{R}\times S^1 \to M \mid \bar\partial_{J,H} u = 0,\ u(s,\cdot) \xrightarrow[s\to\pm\infty]{} x^\pm,\ \text{ind } D_u\bar\partial = k\}/\mathbb{R}$
are smooth manifolds of dimension $k-1$; $k \geq 1$
cut out <u>transversely</u> by Fredholm section $\bar\partial_{J,H} \curvearrowright \mathcal{B} \downarrow W^{1,p}_{x^-,x^+}(\mathbb{R}\times S^1, M)$]

for "generic" $J : S^1 \to \mathcal{J}(M,\omega)$
$= \{\omega\text{-compatible almost complex structures}\}$

- $\{\gamma$ <u>crit. pt.</u>$\} \subset \mathcal{C}^\infty(S^1, M)$ is <u>compact</u> $\underset{\text{Morse}}{\Longrightarrow}$ a finite set

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- trajectories of finite energy connect critical points

$$\left[\ E(u) := \int_{S^1\times\mathbb{R}} |\partial_s u|^2 < \infty \quad\Rightarrow\quad u|_{\{s\}\times S^1} \xrightarrow[s\to\pm\infty]{} x^\pm \in \text{crit} f_H \right.$$

$$\left.\ \ \| \qquad \int \langle \partial_s u, J\partial_t u - \nabla H(u)\rangle = \ldots = f_H(x^+) - f_H(x^-) \mod \langle \omega, \pi_2\rangle \right.$$

$\hat{\mathbb{R}}$

<u>ASSUME MONOTONICITY</u> $[\omega] = \tau \cdot c_1(TM, J)$ on $\pi_2(M)$; $\tau \geq 0$

$$\Rightarrow E(u) = \tau \cdot \text{ind } D_u\bar\partial + E_0 \qquad \forall u$$

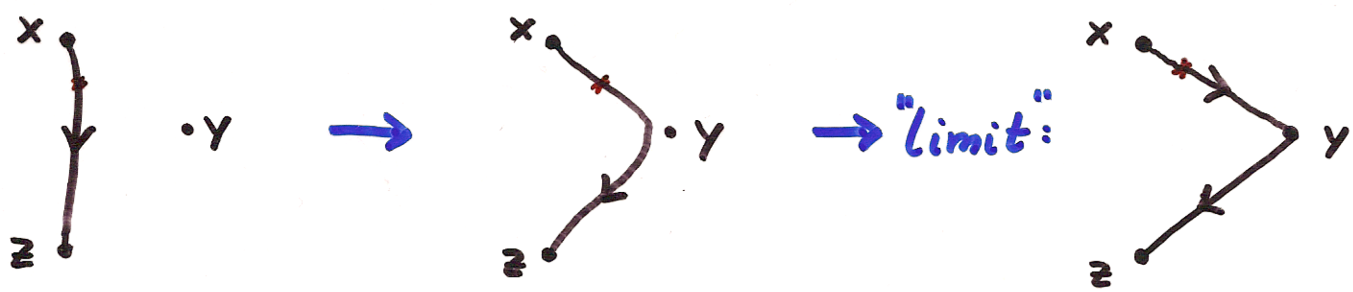
- <u>Gromov compactness</u> : $\mathcal{M}^k$ are compact up to

→ breaking $\int |\partial_s(\lim_{\nu\to\infty} u^\nu)|^2 < \lim_{\nu\to\infty} \int |\partial_s u^\nu|^2$ | → bubbling $\|\partial_s u^\nu\|_\infty \xrightarrow[\nu\to\infty]{} \infty$
$\qquad\qquad \| \ \ \nabla f_H(u^\nu)$

Compactification of moduli spaces

possible divergence of trajectories:

Morse theory:

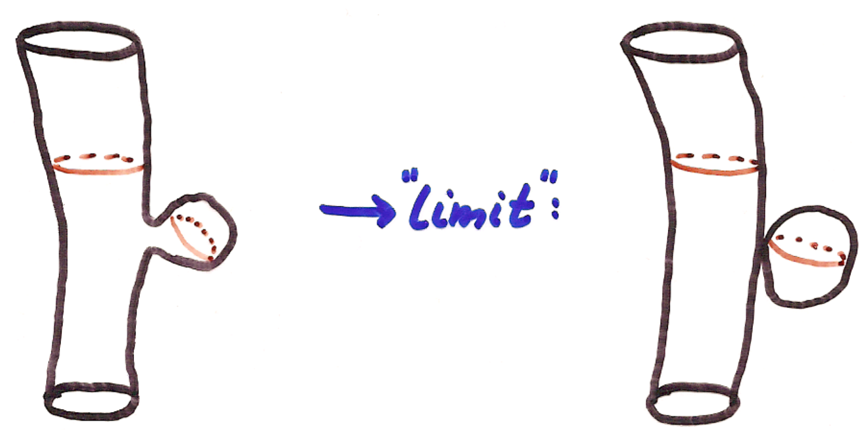


sequence of trajectories → "broken trajectory"

Floer theory:

"broken trajectory"

can exclude bubbling in 0- and 1-dim. moduli spaces (e.g. by monotonicity)



"bubbling"

1. periodic symplectic Floer homology

$\varphi \in \mathrm{Symp}(M,\omega) = \{\varphi : M \xrightarrow{\sim} M \mid \varphi^*\omega = \omega\}$

$\left(\text{e.g. } \varphi_H^1 = \text{time 1 flow of } X_H = -J\nabla H\right.$

$\mathcal{X}_\varphi = \{\gamma : [0,1] \to M \mid \gamma(0) = \varphi(\gamma(1))\}$ $\quad \begin{pmatrix} & \xrightarrow{\sim} & \mathcal{X}_M \\ \gamma & \longmapsto & \hat{\gamma}(t) = \varphi_H^t(\gamma(t)) \end{pmatrix}$

$f_M : \mathcal{X}_\varphi \to \mathbb{R}/_{***}$ symplectic action

$\gamma \longmapsto -\int v^*\omega$ $\qquad v : [0,1]\times[0,1] \to M \quad v(s,0) = \varphi(v(s,1)) \ \forall s$

$v(0,\cdot) = \gamma_0$ fixed (in each connected component of $\mathcal{X}_\varphi$)

$v(1,\cdot) = \gamma$

- **critical points:**

$\boxed{\text{fixed points of } \varphi \ : \ \gamma(t) \equiv \gamma(0) = \varphi(\gamma(0))}$

- **Floer trajectories:**

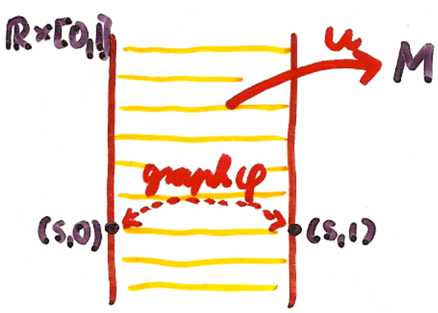
$$\boxed{u : \mathbb{R}\times[0,1] \to M \qquad \begin{cases} \bar\partial_J u := \partial_s u + J(t,u)\,\partial_t u = 0 & \forall s,t \\ (u(s,1), u(s,0)) \in \operatorname{graph}\varphi & \forall s \end{cases}}$$

Thm (Floer, ...): $HM_*(\mathcal{X}_\varphi, f_M) =: HF_*(M,\varphi)$ exists

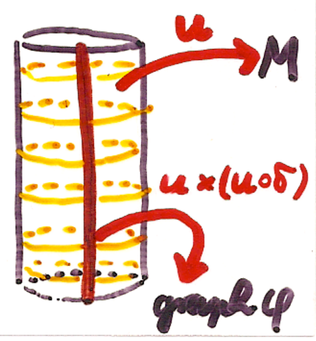
... and only depends on $\varphi \in \mathrm{Symp}(M,\omega)/\mathrm{Ham}(M,\omega)$

$\mathrm{Ham}(M,\omega) = \{\varphi_H^1 \mid H : [0,1]\times M \to \mathbb{R}\}$

Note: trajectories are quilts



- patch $\mathbb{R}\times[0,1]$
- seam $\sigma : (s,1) \mapsto (s,0)$



② instanton Floer homology

$\mathcal{X}_Y = \Omega^1(Y; su(2)) / \mathcal{G}(Y) = \dfrac{SU(2)\text{-connections}}{\text{bundle isomorphisms}}$ on Y 3-manifold

$f : \mathcal{X}_Y \to \mathbb{R}/4\pi^2\mathbb{Z}$ Chern-Simons functional , L^2-metric on $\mathcal{X}_Y$

$\tilde{f} : \Omega^1(Y; su(2)) \to \mathbb{R}$, $\nabla\tilde{f}(A) = *F_A = *(dA + A \wedge A)$

(curvature)

critical points:

flat connections A ($F_A = 0$) / bundle isomorphisms

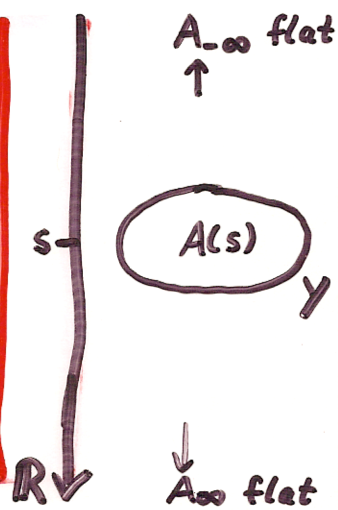
Floer trajectories:

$$\{ A : \mathbb{R} \to \Omega^1(Y; su(2)) \mid \partial_s A + \underbrace{*F_A}_{\nabla f} = 0 \} / \mathcal{G}(Y)$$

$$\cong$$

$$\{ \Xi \in \Omega^1(\mathbb{R} \times Y; su(2)) \mid F_\Xi + *F_\Xi = 0 \} / \mathcal{G}(\mathbb{R} \times Y)$$

anti-self-dual connections (Instantons)



Thm (Floer): Assume $H_*(Y; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$ (or use a nontrivial $SO(3)$-bundle on Y)

then $HM_*(\mathcal{X}_Y, f) =: HF_*(Y)$ exists

..... and satisfies an exact sequence for surgeries on knots $K \subset S^3$.

Kronheimer-Mrowka

Property P : surgery on a knot cannot be a counter example to the Poincaré conjecture

③ Seiberg-Witten Floer homology

Floer theories "with boundary"

1' symplectic Floer homology for Lagrangian submanifolds

$\mathcal{X} = \{\gamma : [0,1] \to M\}$ path space

has no action functional but let's try the "vector field" from the closed case ①, i.e. the 1-form $\lambda = \langle \text{"}\nabla f_{H=0}\text{"}, \cdot \rangle : T\mathcal{X} \to \mathbb{R}$

$$d\lambda(\xi, \zeta) = \omega(\xi(0), \zeta(0)) - \omega(\xi(1), \zeta(1))$$

ω : symplectic 2-form on M : $d\omega = 0$, nondegenerate
(i.e. $\omega(\cdot, J\cdot)$ is a metric)

So λ is a closed form on the path space

$\mathcal{X}_{M,L_0,L_1} = \{\gamma : [0,1] \to M \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}$

with Lagrangian boundary conditions

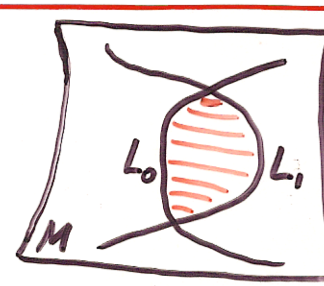
$L_i \subset (M, \omega)$ submanifold, $\omega|_{TL} \equiv 0$, $\dim L = \frac{1}{2} \dim M$ (maximal)

Thm (Floer, ...): Assume ~~[illegible]~~ (or $\pi_2(M, L_i) = \{0\}$),
then $HM_*(\mathcal{X}_{M,L_0,L_1}, \lambda) =: HF_*(M; L_0, L_1)$ exists.

critical points:
$L_0 \cap L_1$

Floer trajectories:
$u : \mathbb{R} \times [0,1] \to M$ $\begin{cases} \partial_s u + J \partial_t u = 0 \\ u|_{t=i} \in L_i \end{cases}$

pseudoholomorphic curves



2' instanton Floer homology with Lagrangian boundary conditions

Thm (Salamon, W.): Y compact 3-manifold with boundary $\partial Y = \Sigma$

$\mathcal{L} \subset \Omega^1(\Sigma; su(2))$ Lagrangian submanifold (gauge invariant, ~~[illegible]~~)

$HM_*(\mathcal{X}_{Y,\mathcal{L}}, \lambda_{cs}) =: HF_*(Y, \mathcal{L})$ exists (and is independent of the metric on Y)

trajectory PDE	Floer homology invariant	book / script
J-holomorphic cylinder	$HF(H) \cong H_*(M)$	Salamon (Park City)
— " — cylinder w. twist	$HF(\varphi$ symplectomorphism$)$	
strip w. boundary cond.	$HF(L_0, L_1$ Lagrangians$)$	Seidel Fukaya-Oh-Ohta-Ono
quilt w. seam cond.	$HF(\underline{L}$ cyclic Lagrangian correspondence$)$	[W-Woodward]
anti-self-dual Yang-Mills	$HF(Y$ 3-manifold$)$	Donaldson
Seiberg-Witten	$\widehat{\check{HF}}^{\infty}(Y$ 3-manifold$)$ $\Vert$ conj.	Kronheimer-Mrowka
J-hol. strip & Heegard splitting	$\widehat{\check{HF}}^{\infty}(Y$ 3-manifold$)$ $\cong$	Ozsvath-Szabo
$Y = H_\alpha \cup_\Sigma H_\beta$	$HF(T_\alpha, T_\beta \subset Sym^g \Sigma)$	
		= Atiyah-Floer conjecture
	${}^?HF^?(Y$ 3-manifold$)$ $\Vert$	
	${}^?HF^?\left(L_{H_\alpha}, L_{H_\beta} \subset \mathcal{H}om(\pi_1(\Sigma), SU(2))/SU(2)\right)$ $\Vert$ conj.	
	$HF\left([0,1]\times\Sigma; \mathcal{L}_{H_\alpha}, \mathcal{L}_{H_\beta} \subset \Omega^1(\Sigma; su(2))\right)$	Salamon-W.
J-hol. quilts & 3-mfd decomposition	$HF(Y = Y_{01} \cup \ldots \cup Y_{(k-1)k}) = HF(L_{Y_{01}}, \ldots, L_{Y_{(k-1)k}})$	[W.-Woodward]