

# Cornell Topology Festival Panel Discussion

May 2010

On May 9th, 2010, the speakers from the Cornell Topology gave a panel discussion on recent developments (other than their own work) and interesting questions in topology and related fields. The speakers were asked to limit their presentations to 5 minutes. Notes were taken and compiled by members of the Cornell Bernstein seminar. Any errors in what follows are almost certainly due to us, and not the speakers.

## Miklos Abert (Chicago)

Abert spoke about the history of the Burnside Problem and related questions. This very early question in group theory asks: “Is there a finitely generated infinite torsion group?”. Golod and Shafarevich were the first to provide examples of such groups, over 50 years ago. Grigorchuk produced his eponymous example group, and other automata group theorists have produced more examples. There is now a very easy answer to the question, which would take just slightly more time than the 5 minute presentation to relate.

Abert next discussed the question: “Must residually finite nonamenable groups contain nonabelian free groups?”. Ershov showed that every Golod-Shafarevich group has a quotient with Kazhdan’s property (T), thereby answering this question in the negative. Osin produced further counterexamples, using positivity of the rank gradient to show they were nonamenable. Schlage-Puchta has a 3 page construction of another counterexample, a finitely generated, residually finite  $p$ -group with positive rank gradient.

Abert observed that none of the counterexamples for the second question are finitely presented, so he poses the modified question: “Is there a residually finite, *finitely presented*, nonamenable group that does not contain a nonabelian free group?” He speculated on the utility of thinking about a mythical “finite ping pong lemma” to motivate a (negative) solution.

## Indira Chatterji (Ohio State University and Université d’Orléans)

Thompson’s group  $F$  is the group of piecewise-linear homeomorphisms of the unit interval where

1. all slopes are powers of 2, and
2. there are finitely many breakpoints, which all occur at dyadic rationals.

The group  $F$  is finitely presented, torsion-free, has exponential growth, and does not contain non-abelian free groups.

A year ago E.T. Shavgulidze ([1]) put forward a proof showing Thompson’s group  $F$  to be amenable. Since then a gap has been discovered. This short talk is about a results contained in Shavgulidze’s paper, which I’m not sure is correct, but if correct then very interesting.

Consider the group  $\text{Diff}_+^3([0, 1])$  of all  $C^3$ -diffeomorphisms of  $[0, 1]$  fixing the endpoints of the interval. Let  $\text{Diff}_0^3([0, 1]) < \text{Diff}_+^3([0, 1])$  be the subgroup of diffeomorphisms with right derivative at 0 and left derivative at 1 both being 1.

A subgroup  $G < \text{Diff}_0^3([0, 1])$  satisfies condition A if there exists a  $C > 0$  such that for every  $g_1 \neq g_2$  in  $G$ ,

$$\sup_{t \in [0, 1]} |\ln(g_1'(t)) - \ln(g_2'(t))| \geq C$$

**Theorem 1** (Shavgulidze). *Any  $G$  satisfying condition A is amenable.*

## Karsten Grove (Notre Dame)

Grove began by giving a geometric description of positive sectional curvature, and discussed briefly the classical Synge method for obtaining topological results from positive sectional curvature. Grove then spoke of a new method due to Wilking, which gives, for example, the following:

**Theorem 2** (Wilking). *Suppose  $M$  has positive sectional and  $i : V^n \rightarrow M^{n+k}$  an embedding so that  $i(V)$  is totally geodesic. Then  $i$  is  $n - k + 1$  connected.*

Grove then posed the following question:

**Question 1.** *Suppose  $C = M_0 \subset M_1 \subset M_2 \subset \dots$  is an infinite chain of manifolds each with positive sectional curvature, each  $M_i$  totally geodesic in  $M_{i+1}$ . Does the chain have to be one of the following?*

- $S^n \subset S^{n+1} \subset \dots$
- $\mathbb{C}\mathbb{P}^n \subset \mathbb{C}\mathbb{P}^{n+1} \dots$
- $\mathbb{H}\mathbb{P}^n \subset \mathbb{H}\mathbb{P}^{n+1} \dots$

## Dan Margalit (Tufts)

Let  $f : S \rightarrow S$  be a pseudo-Anosov map, and  $\lambda(f) = \lambda$  the dilatation of  $f$ .

**Question 2.** *What are the possible values of  $\lambda$ ?*

McMullen has conjectured that  $\lambda$  is a dilatation if and only if  $\lambda$  and  $\frac{1}{\lambda}$  are Perron numbers. (A Perron number  $\alpha$  is an algebraic integer which is real, greater than 1, and all its galois conjugates have norm less than  $\alpha$ .)

**Question 3.** *What is the minimal  $\lambda$  for a given surface?*

The minimal dilatation  $\lambda$  has been determined for about 9 surfaces.

**Question 4.** *What does a small pseudo-Anosov look like?*

An easier question is to restrict to a subclass of homeomorphisms. Let  $S$  be a surface with a distinguished point  $p$ , and  $Mod(S, p)$  be the mapping class group. Then  $Mod(S, p)$  is the group of homeomorphisms of  $S$  fixing the point  $p$  up to isotopy fixing  $p$ . The push map is the obvious map  $\pi_1(S, p) \rightarrow Mod(S, p)$ .

**Theorem 3** (Kra). *If  $\gamma$  is filling then  $push(\gamma)$  is pseudo-Anosov.*

**Theorem 4** (Spencer Dowdall). *Let  $\lambda_\gamma$  be the dilatation corresponding to  $push(\gamma)$  and  $i(\gamma)$  denote the self intersection number of  $\gamma$ . Then,*

$$\sqrt[5]{i(\gamma)} \leq \lambda_\gamma \leq 9^{i(\gamma)}.$$

Note that these bounds are independent of the surface.

## Jeremy Kahn (Stony Brook)

**Question 5.** *Can we find a surface subgroup of the mapping class group  $MCG(S_g, 1)$  with all pseudo-Anosovs?*

One possible approach: Let  $M$  be a 3-manifold which fibers over the circle. Let  $F$  denote the fiber with distinguished point  $\star$ , and let  $S$  be an immersed surface transverse to the fiber.

$$\begin{array}{ccccc}
 & & F \times \mathbb{R} & & \\
 & \nearrow & \downarrow & & \\
 F & \longrightarrow & M & \longleftarrow & S \\
 & & \downarrow & & \\
 & & S^1 & & 
 \end{array}$$

We get a push map  $p_s : \pi_1(S) \rightarrow MCG(F, \star)$ . (Since  $M$  fibers over the circle, the entire fundamental group of  $\pi_1(M)$  is contained in  $MCG(F, \star)$ .)

If for all geodesics on  $S$  such that  $\gamma$  lifts to  $F \times \mathbb{R}$ ,  $\gamma$  fills  $F$ , then  $p_S(\pi_1(S))$  has all pseudo-Anosovs. Suppose  $\gamma$  lifts to  $F \times \mathbb{R}$ . The width of  $\gamma$ ,  $w(\gamma)$ , is the width of the projection of the lift of  $\gamma$  to  $\mathbb{R}$ .

**Theorem 5** (Richard Kent). *For all  $M$  fibering over  $S^1$ , there exists a  $w_M$  such that if  $\gamma$  lifts to  $F \times \mathbb{R}$  and  $w(\gamma) > w_M$ , then the lift of  $\gamma$  fills  $F$ .*

This result reduces the problem to finding an  $S$  transverse to the fibers of  $M$  so that all  $\gamma$  that lift have large width.

## Nikolay Nikolov (Imperial College London)

We discuss a problem posed by Barnea, Ershov, Weigel and Klopsch.

**Question 6.** *Which profinite groups can be open compact subgroups of a simple, locally compact, totally disconnected group?*

The notation for this will be  $U <_0 G$ , where  $U$  is the profinite group and  $G$  the simple l.c.t.d. group. Since two different such embeddings of  $U$  in  $G$  are commensurable (their intersection is finite index in both), this question is related to the study of the commensurizer of the profinite group  $U$ .

An example is  $SL_n(\mathbb{Z}_p) \subset SL_n(\mathbb{Q}_p)$ . Also, the Grigorchuk group can be embedded as an open compact subgroup of simple l.c.t.d. group. It is unknown for the free pro- $p$  group on 2 generators.

**Lemma 1.** *If  $U <_0 G$  for  $G$  compactly generated, then there is a finite set of primes that divide the orders of the finite continuous quotients of  $U$ .*

This can be used to rule out  $\Pi_{n \geq 5} A_n$ . There is also a negative answer for solvable profinite groups, by a result of G. Willis.

## Ed Swartz (Cornell)

A graph is *planar* if it can be embedded in the Euclidean plane. Given a graph  $G$ , a graph  $H$  is a *minor* of  $G$  if there is a subgraph  $G' < G$  such that  $H$  can be obtained from  $G'$  by one or more edge contractions.

**Theorem 6** (Kuratowski's Excluded Minors Theorem). *A graph is planar iff it does not contain a complete graph on 5 vertices or a complete bipartite graph on 3 and 3 vertices as a minor.*

**Question 7.** *What about embedding in surfaces other than a plane? Is there an analogue of Kuratowski's theorem? That is, can a finite collection of excluded minors always be specified?*

The answer is yes!

**Theorem 7** (Robertson-Seymour). *There does not exist an infinite collection of graphs in which no graph is a minor of another.*

However, there is no efficient algorithm for finding the excluded minors for a given surface. For example, there are more than 1000 known excluded minors for embeddability in a torus.

A simplicial complex  $\Delta$  is *pure* if every maximal simplex has the same dimension. A *matroid* is a simplicial complex in which all vertex induced subcomplexes are pure. For example, let the vertices be a collection of vectors (in any vector space) and simplices be independent subsets.

A matroid is *representable* over a field  $\mathbb{F}$  if it has the above form. That is, its vertices correspond to vectors (in  $\mathbb{F}^n$  for some  $n$ ) and simplices are linearly independent subsets. A matroid  $N$  is a *minor* of matroid  $M$  if there is a submatroid  $M' < M$  such that  $N$  can be obtained from  $M'$  by a sequence of (simplicial) contractions.

**Question 8** (Rota). *Given a finite field  $\mathbb{F}$ , is there a finite set of excluded minors for matroids representable over  $\mathbb{F}$ ?*

This has been answered (affirmatively) for the fields of 2, 3, and 4 elements. The general question is still open. One way to answer this question is to prove a counterpart of the Robertson-Seymour theorem for matroids.

## Daniel Wise (McGill)

Mark Hagen, a student of Daniel Wise, discussed recent results on asymptotic dimension of metric spaces. Consider a metric space  $M$ . We say  $asdim M \leq n$  if  $\forall r > 0 \exists \{U_\alpha\}$  with  $\cup_\alpha U_\alpha = M$  such that at most  $n + 1$  of the  $U_\alpha$ 's meet a ball of radius  $r$ . For example a tree  $T$  has  $asdim T \leq 1$ . Some groups that were shown to have finite asymptotic dimension include:

1. hyperbolic
2.  $A *_C B$ , where  $A, B$  are finite (by Bell and Dranishnikov)
3. relatively hyperbolic (by Osin)

A theorem recently proved by Nick Wright states: Let  $\tilde{X}$  be  $CAT(0)$  cube complex with  $\dim \tilde{X} = D < \infty$ , then  $asdim \tilde{X} \leq D$ .

## Robert Young (IHES)

Robert Young discussed the work of Stefan Wenger on the growth rate of Dehn functions of nilpotent groups. Dehn functions measure the complexity of the word problem in a given finitely presented group and provide a quasi-isometry invariant of the group. It has been prior proved by Sapir and Olshanski that for certain class of groups,  $\delta_G(n) \lesssim n^2 \log n$ . In his recent work, Stefan Wenger proved that there exist finitely generated nilpotent group  $G$ , such that  $n^2 \not\lesssim \delta_G(n) \lesssim n^2 \log n$ . Thus,  $G$  has a Dehn function which is not equivalent to a polynomial.

## References

- [1] E.T. Shavgulidze, The Thompson group  $F$  is amenable, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 12, 2009, no. 2, 173–191.