Manifistations of the Lamplighter

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The main HERO

$\mathcal{L} = \mathbb{Z}_2 \wr \mathbb{Z}$ – Lamplighter

$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

His "command"

$F \wr G$, $|F| < \infty$, $|G| = \infty$

Other personages: iterated wreath products.

I. The permutational wreath product.

A, G two groups, $\quad A \neq \{e\}$

$G \circlearrowright X \qquad G$ acts on X

$$W = \Big(\prod_X^* A\Big) \rtimes G = A \wr G \text{ - restricted wreath product}$$

$\prod_X^* A$ - weak direct product (only finitely many entries are $\neq \{e\}$)

$G \underbrace{\text{act on}}_{\text{by automorphisms}} \prod_X^* A$ on the left by $(g \cdot f)(x) = f(xg)$.

$$A \bar{\wr} G = \left(\prod_X A\right) \rtimes G$$ – unrestricted wreath product

$Wr = \bar{\wr}$ $\qquad$ $wr = \wr$

$$|A|, |B| < \infty \Rightarrow A \wr B = A \bar{\wr} B$$

Examples: $\mathcal{L} = \mathbb{Z}_2 \wr \mathbb{Z}$ The Lamplighter

$|F| \wr G$, $G = \mathbb{Z}^d$, F_d, $B(m,n)$ – free Burnside gp, group of intermediate growth

$\mathbb{Z} \wr \mathbb{Z}$, iterated wreath products

A, G, S_A, S_G - systems of generators for A and G

$S = S_A \sqcup S_G$ - system of generators for $A \wr G$

$$G \ni g \longrightarrow (1, g)$$

$$A \ni a \longrightarrow (f_a, 1)$$

embeddings

$f_a(x_0) = a$, $f_a(x) = e$, $x \neq x_0$

$x_0 \in X$ - distinguished point

A and B are finitely generated $\Rightarrow$ $A \wr B$ is f.g.

G. Baumslag. $A \wr B$ is not finitely presentable if $A \neq \{e\}$ and $|B| = \infty$

$\Rightarrow$ the Lamplighter is not finitely presentable

$$\mathcal{L} = \langle a, b \mid a^2 = 1, [a, a^{b^n}] = 1, n = 1, 2, \dots \rangle$$

$$x^y = y^{-1} x y \qquad [x, y] = x^{-1} y^{-1} x y$$

Th. [Baumslag, Remeslennikov] Every finitely generated metabelian group embeds into a finitely presented metabilian group.

Apply this to the Lamplighter.

$a = (\cdots 0, 1, 0, \ldots) \in \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$ – base group

b – a generator of active group $\mathbb{Z}$.

Th. [Baumslag]. Let $\alpha : \mathcal{L} \to \mathcal{L}$ be given by $\alpha(a) = [a, b]$, $\alpha(b) = b$. This defines an injective group homomorphism, and

$$G = \langle a, b, s \mid a^2 = [b, s] = [b^{-1}ab, a] = e,\ s^{-1}as = [a, b] \rangle$$

is isomorphic to the ascending HNN extension of $\mathcal{L}$ along α.

$\mathcal{L} \hookrightarrow G$ is Baumslag–Remeslennikov type embedding

[will be used for Atiyah Problem].

Growth

G - finitely generated group

A - finite system of generators

$|g| = |g|_A$ - the length of g w.r.to A

$$\gamma(n) = \gamma_G^A(n) = \#\{g \in G \mid |g| \leq n\}$$

$$= \# B_e(n)$$ - ball of radius n in Cayley graph

$\gamma_G^A(n) \sim \gamma_G^B(n)$

Milnor equivalence

$\left[\gamma_1(n) \sim \gamma_2(n) \iff \exists\, c \text{ s.t. } \gamma_1(n) \le c\,\gamma_2(cn), \; \gamma_2(n) \le c\,\gamma_1(cn)\right]$

$\gamma_1(n) \preceq \gamma_2(n)$

partial order

$[\gamma(n)]$ – the growth degree of G

class of equivalence

$$\Gamma(z) = \Gamma_G^A(z) = \sum_{n=0}^{\infty} \gamma(n) z^n$$ – growth series

$\Gamma(z)$ rational or even algebraic $\Rightarrow$ $\gamma(n) \sim P(n)$ or $\gamma(n) \sim 2^n$

polynomial

exponential

Th. [W. Parry 92]. For $G = F \wr F_m$, $m \geq 2$ the growth series is an algebraic irrational function.

Growth can be polynomial, intermediate (between polynomial or exponential), and exponential

Th. (GRi) 1984. 1) There are uncountably many 2-generated groups of intermediate growth.

2) The partially ordered set of growth degrees contains a chain of the cardinality of the continuum and contains an antichain of the cardinality of the continuum.

Corollary. Up to quasi-isometry there are uncountably many 2-generated groups.

The main example. $G = \langle a, b, c, d \rangle \curvearrowright X = [0,1] \setminus Q_2$

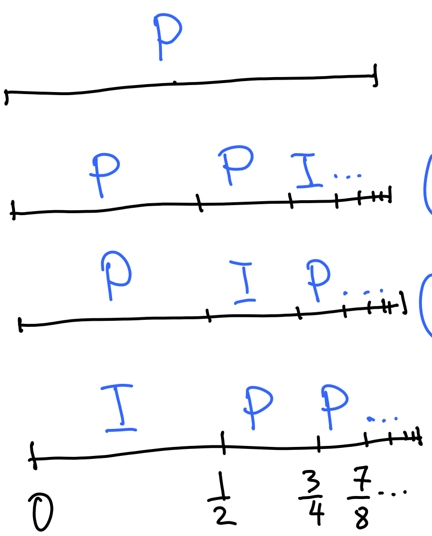
diadic rational points

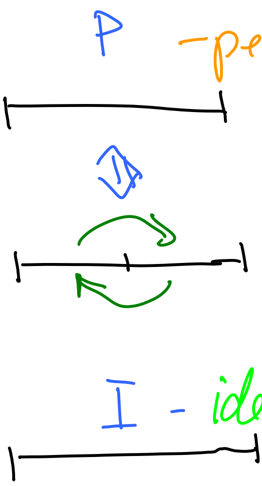
a

b $(PPI)^\infty$

c $(PIP)^\infty$

d $(IPP)^\infty$





P – permutation of two halves of the interval

I – identity transformation

Th. (Gri. 84) $e^{\sqrt{n}} \leqslant \gamma(n) \leqslant e^{n^{\beta}}, \quad \beta = \log_{32} 31 < 1$

Y. Leonov, L. Bartholdi

$\frac{1}{2} \to \frac{1}{2} + 0.4, \quad \beta \to 0.74\ldots$

Bartholdi, Muchnik, Pak

General construction

$\underline{\Omega} = \{0,1,2\}^{\mathbb{N}} \ni \omega = \omega_1 \omega_2 \ldots\ldots$ – infinite sequences

$0 \longleftrightarrow \begin{pmatrix} P \\ P \\ I \end{pmatrix} \qquad 1 \longleftrightarrow \begin{pmatrix} P \\ I \\ P \end{pmatrix} \qquad 2 \longleftrightarrow \begin{pmatrix} I \\ P \\ P \end{pmatrix}$

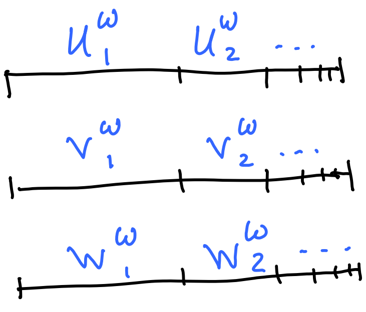
$$\omega \longleftrightarrow \begin{pmatrix} U_\omega \\ V_\omega \\ W_\omega \end{pmatrix} = \begin{pmatrix} u_1^\omega & u_2^\omega & \ldots & u_n^\omega & \ldots \\ v_1^\omega & v_2^\omega & \ldots & v_n^\omega & \ldots \\ w_1^\omega & w_2^\omega & \ldots & w_n^\omega & \ldots \end{pmatrix}$$

$3 \times \infty$ matrix

b_ω

c_ω

d_ω



$G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$

$G = \langle a, b, c, d \rangle = G_{(012)^\infty}$

$G_{(01)^\infty}$ – also important gp

$$\Omega = \Omega_1 \sqcup \Omega_2$$

$\Omega_2 = \{\omega \mid \omega$ is constant at $\infty\}$

$\omega = \omega_1 \dots \omega_n\, i\, i\, i \dots i \dots, \quad i \in \{0,1,2\}$

T1. 1) $\omega \in \Omega_1 \Rightarrow G_\omega$ is of intermediate growth and is branch just-infinite group

2) $\omega \in \Omega_2 \Rightarrow G_\omega$ is virtually abelian group.

The "surgery" of the construction: Delete from $\{G_\omega\}_{\omega\in\Omega}$ the countable set $\{G_\omega\}_{\omega\in\Omega_2}$ and take the closure in the space of marked 4-generated groups.

We replace G_ω, $\omega \in \Omega_2$ by $\tilde{G}_\omega$ - virtually metabelian groups of exponential growth and get a Cantor set of groups.

$$\tilde{G}_{0^\infty} \simeq \mathcal{L} \rtimes \mathbb{Z}_2 \simeq G(A_{891})$$

[Bondarenko, GRi, Kravchenko, Nekrashevych, Muntyan, Savchuk, Sunik. Atlas of self-similar groups]

The presentence of the Lamplighter in this construction allowed to get uncountably many growth degrees.

$$e^{n^{\frac{1}{2}+0.04}} \preceq \gamma_{G_{(012)^\infty}}(n) \preceq e^{n^\beta}$$

Q. Is $\gamma_{G_{(012)^\infty}}(n) \sim e^{n^\delta}$ for some δ? What is δ?

$\forall \varepsilon > 0$

$$e^{\frac{n}{\log^{2+\varepsilon} n}} \preceq \gamma_{G_{(01)^\infty}} \preceq e^{\frac{n}{\log^{1-\varepsilon} n}}$$

A. Erschler 2004 Annal of Math.

Q. Is $\gamma_{G_{(01)^\infty}}(n) \sim e^{\frac{n}{\log^\delta n}}$ for some δ, $1 \le \delta \le 2$? What is δ?

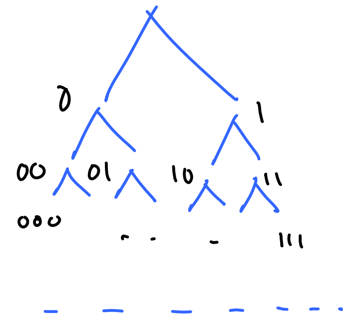
Th [L. Bartholdi and A. Erschler 2010] There are two infinite sequences of groups $\{G_k\}_{k=1}^{\infty}$ and $\{H_k\}_{k=1}^{\infty}$ and a sequence of positive numbers $\alpha_k = 1-(1-\alpha)^k$ where $\alpha = \log 2/\log\left(\frac{2}{\eta}\right) \approx 0.764..$ and η is a root of X^3+X^2+X-2 s.t.

$$\gamma_{G_k}(n) \sim e^{n^{\alpha_n}}$$

$$\gamma_{H_k}(n) \sim e^{(\log n)\, n^{\alpha_n}}$$

The construction is based on the use of permutational wreath product and the notion of $\boxed{\text{inverted orbit growth}}$

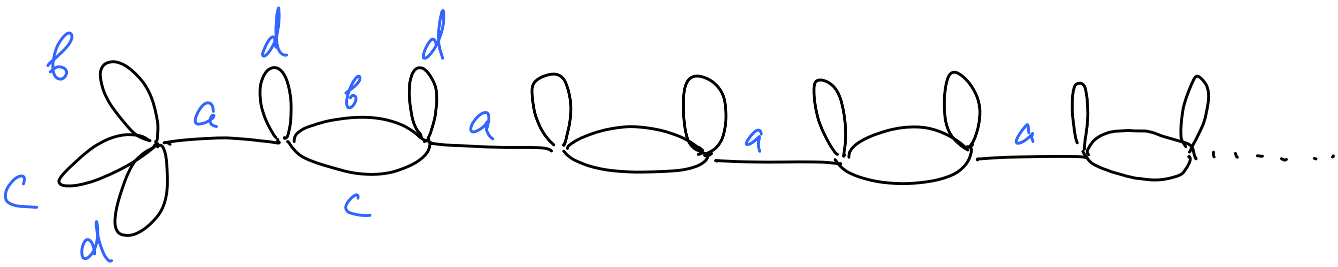
$G = \langle a, b, c, d \rangle = G_{(012)^\infty}$ – act on a binary rooted tree



$P = St_G(1^\infty)$ – stabilizer

$\Gamma = \Gamma(G, P, \{a,b,c,d\})$ – Schreier graph

$1^\infty \in \partial T = \{0,1\}^{\mathbb{N}}$ – Boundary



Γ has linear growth $\sim n$ but sublinear inverted growth of the type n^α, $\alpha = 0.764\ldots$

$G \curvearrowright X$ (right action) $X \ni x_0$ - distinguished point

inverted orbit of a word $w = g_1 \dots g_n$ over G is

$$\{x_0 g_1 \cdots g_n,\ x_0 g_2 \cdots g_n,\ \dots,\ x_0 g_{n-1} g_n,\ x_0 g_n\} = IO_{x_0}(w)$$

$$\delta(w) = \# IO_{x_0}(w)$$

inverted orbit growth function is

$$\Delta(n) = \max\{\delta(w) \mid |w| = n\}.$$

Th. Let A be a non-trivial group having growth $\sim e^{\frac{n}{\varphi(n)}}$, and assume $\frac{n}{\varphi(n)}$ is convex. Consider the wreath product $M = A \wr_X G_{(012)^\infty}$. Then the growth of M is $\sim e^{\frac{n}{\varphi(n^{1-\alpha})}}$.

Bartholdi, Erschler

Amenability

J. von Neumann (discrete case) 1929

L. Alphors (amenable Riemannian surfaces) 1935

N. N. Bogolyubov (general topological groups) 1939

Def. [A group is amenable if it does not allow Ponzi scheems]

1) G is amenable if it has LIM (left invariant mean)

i. e. a finitely additive left invariant measure μ defined on the σ-algebra of all subsets of G with values in $[0,1]$ and normalized by $\mu(G)=1$.

2) A group G is superamenable if for every action $G \curvearrowright X$ and nonempty subset $A \subset X$ there is a finitely additive measure μ defined on the σ-algebra of all subsets of X with values in $[0, +\infty]$ normalized by the condition $\mu(A) = 1$.

superamenable $\Rightarrow$ amenable $\not\Rightarrow$ superamen.

$F_2 \hookrightarrow G \Rightarrow G$ is non amenable

free group

$FS_2 \hookrightarrow G \Rightarrow G$ is not superamenable.

G is amenable $\Leftrightarrow$ $\forall$ action on a compact space X there is a G-invariant probabilistic measure on X (Bogolyubov 1939)

Day 50th

G is superamenable $\iff$ $\forall$ action on a topological space there is an invariant Radon measure.

J. Rosenblatt introduced the notion of superamenable gp, proved that subexponential growth $\Rightarrow$ superamenability and Conjectured. A group is superamenable if and only if it is amenable and does not contain a free semigroup FS_2 on two generators.

GRi in 1987 gave a counter-example.

Th. Let $G = \langle a, b, c, d \rangle$ be a 2-group of intermediate growth and $L = \mathbb{Z}_2 \wr G$. Then L is torsion amenable group (and hence does not contain FS_2) but is not superamenable.

It was showed that instead of SF_2 the Cayley graph of L contains a paradoxical binary rooted tree.

Two related problems of Rosenblatt.

Problem. Does any group G of exponential growth admit a Lipschitz imbedding of the infinite binary tree?

Problem. Is every superamenable group exponentially bound? (i.e. of subexponential growth).

Fölner criterion and Fölner function

A finitely generated group G is amenable $\Longleftrightarrow$

$$\inf_{\substack{E \subset V(\Gamma) \\ |E| < \infty}} \frac{|\partial E|}{|E|} = 0 .$$

$\Gamma = \Gamma(G, A)$ – Cayley graph, ∂E – boundary of a subset $E \subset V(\Gamma)$

$$F(r) = \text{Föl}(r) = \min\left\{ |E| : E \subset V(\Gamma), \frac{|\partial E|}{|E|} < \frac{1}{r} \right\},$$

$r \in (1, +\infty)$

A. Vershik, 70th.

Q. How $F(r)$ grow as $r \to +\infty$?

Varopoulos, Coulhon and Saloff-Coste, Pittet and Saloff-Coste:

1) G is virtually nilpotent with polynomial growth of type n^d, then $F(r) \sim r^d$!

2) Fölner function of $\mathbb{Z}_K \wr \mathbb{Z}^d$, $d \geq 2$ is super-exponential

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Th. [A. Erschler]. There exists $C>0$ such that the following holds. Let A and B be two finitely generated amenable groups ($|A| \geq 2$). Let S_A and S_B be finite generating sets of A and B respectively. Then

$$F_{A \wr B}(r) \geq c\left(F_A(cr)\right)^{F_B(cr)}$$

$\Rightarrow$ $F_{\mathbb{Z} \wr \mathbb{Z}}(r) \sim r^r$, $F_{\mathbb{Z}_k \wr \mathbb{Z}^d} \sim e^{n^d}$

super exponential growth

$F(r)$

$F \wr G$, F finite, G of polynomial growth d — e^{r^d}

$H \wr G$, H infinite and of polynomial growth, G of polynomial growth d — $e^{r^d \log r}$

$F \wr (\cdots (F \wr (F \wr \mathbb{Z}))\cdots)$, $|F| < \infty$, k-times iterated wreath product — $e^{e^{\cdot^{\cdot^{\cdot^{e^r}}}}} = \exp_k(r)$

$\mathbb{Z} \wr (\ldots (\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})) \ldots)$

k times iterated wreath product, $k \geq 2$

$\exp_{(k)}(r \log r)$

The Dixmier Problem

G is said to be unitarisable if every uniformly bounded representation $\pi : G \to B(\mathcal{H})$ is unitarisable
Hilbert space

i.e. there is an invertible operator S on $\mathcal{H}$ s.t.
$S^{-1}\pi(\cdot)S^{-1}$ is a unitary representation.

Dixmier 1950: amenable $\Rightarrow$ unitarizable

$\Leftarrow$? – Dixmier Problem.

$F_2 \hookrightarrow G \Rightarrow G$ is not unitarizable

G. Pisier

D. Osin, N. Monod and N. Ozawa produced examples of non-unitarisable groups without free subgroups.

Th. [Monod and Ozawa] For any group G, the following assertions are equivalent.

(i) The group G is amenable.

(ii) The wreath product $A \wr G$ is unitarizable for all abelian groups A.

(iii) The wreath product $A \wr G$ is unitarisable for some infinite group A.

$\Rightarrow$ Th. [Monod & Ozawa] The free Burnside group $B(m, np)$, $m, n \geq 2$, $p \geq 665$, n and p odd of exponent np is non-unitarizable.

Random walks

G, μ - symmetric measure on G, $\mu(g) = \mu(g^{-1})$
(say with a finite support)

assume that supp μ generate G.

μ defines a random walk on G:

start at e, $g \xrightarrow{\mu(h)} gh$ - transitions

Right convolution with μ defines a self-adjoint Markov operator $R_\mu : \ell^2(G) \to \ell^2(G)$

$$(R_\mu f)(x) = \sum_{g \in G} f(xg)\mu(g)$$

Harmonic functions and Poisson – Furstenberg boundary.

$M = R_\mu$ – Markov operator $\Delta = id - M$ – Laplace operator

A function f on G is said to be harmonic iff $Mf = f$ (or $\Delta f = 0$).

Poisson – Furstenberg Boundary

There is a G-space (X, ν) with ν being μ-stationary ($\mu * \nu = \nu$) s.t. $\forall$ bounded harmonic function $f(x)$ on G has a (unique) presentation

$$f(g) = \int_X \varphi(gx)\, d\nu(x)$$

$\varphi \in L^\infty(X, \nu)$

Poisson boundary is trivial $\Leftrightarrow$ $\forall$ bounded harmonic function is constant (Liouville Property)

Th. [Kaimanovich and Vershik 82]

1) The Poisson boundary of $F \wr \mathbb{Z}^d$, $d=1,2$, $|F|<\infty$ is trivial

2) The Poisson boundary of $F \wr \mathbb{Z}^d$, $d \geq 3$, $|F|<\infty$, $F \neq \{e\}$ is nontrivial

3) [A. Erschler] When $d \geq 5$ the Poisson boundary is equal to the space of limit configurations

The return probabilities

$$p(t) = P^{(t)}_{e,e} = \mu^{(t)}(\{e\}) = \langle R^t_\mu(\delta_e), \delta_e \rangle_{\ell^2(G)}$$

δ_e - delta function

Laplace operator $\quad \Delta = id - R_\mu$

The L^2- isometric profile $\quad \Lambda : [1, +\infty) \to (0, \infty)$

$$\Lambda(x) = \inf_{1 \le |\Omega| \le x} \lambda_1(\Omega), \qquad \lambda_1(\Omega) = \inf_{\substack{\mathrm{supp} f \subset \Omega \\ \ne \emptyset}} \frac{\langle \Delta(f), f \rangle}{\|f\|_2^2}$$

Λ is a decreasing right-continuous step function

$\Delta, R_\mu \in \mathcal{N}(G)$ - von Neumann algebra generated by right regular representation

$tr_G : N(G) \to \mathbb{C}$ - von Neumann trace:

$$tr_G(A) = \langle A(\delta_e), \delta_e \rangle_{\ell^2(G)}$$

For a self-adjoint operator $A \in \mathcal{N}(G)$ the spectral projection $E_\lambda^A = \chi_{(-\infty,\lambda)}(A) \in \mathcal{N}(\Gamma)$.

The spectral distribution of Δ is $N : [0,\infty) \to [0,\infty)$

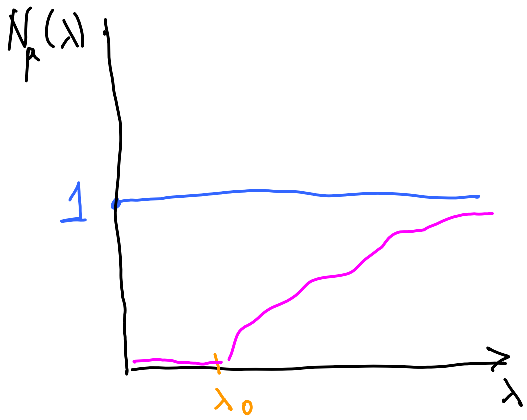
$$N(\lambda) = tr_G(E_\lambda^\Delta) \quad \text{(spectral measure)}$$

The group G is non-amenable if and only if one (then all) of the following assertions holds:

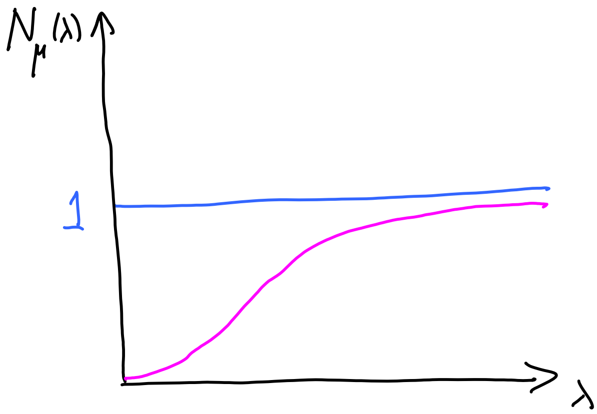
(1) Near infinity, $p_\mu(2t) \sim e^{-t}$ (H. Kesten)

(2) Near infinity, $\Lambda_\mu(x) \sim 1$ ($\Leftrightarrow \exists\, c>0$ s.t. $\Lambda_\mu(x) \geq c \;\; \forall x$)

(3) There exists $0 < \lambda_0$ s.t. $N_\mu(\lambda) = 0$ for all $0 \leq \lambda < \lambda_0$



non-amenable case



amenable c

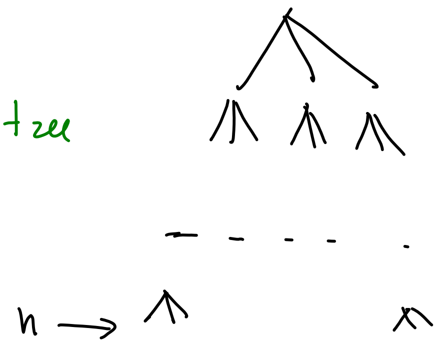
A. Bendikov, C. Pittet and R. Sauer

	$P(2t)$ $t\to\infty$	$N(\lambda)$ as $\lambda\to 0$	$\Lambda(x)$ as $x\to\infty$
$F \wr G$, $\lvert F\rvert<\infty$, G – polyn. deg. d	$e^{-t^{\frac{d}{d+2}}}$	$e^{-\lambda^{-\frac{d}{2}}}$	$(\log x)^{-\frac{2}{d}}$
$H \wr G$, $\lvert H\rvert=\infty$, H – of polyn. gr-th, G – of polyn. g-th d	$e^{\left(-t^{\frac{d}{d+2}}(\log(t))^{\frac{2}{d+2}}\right)}$	$e^{-\lambda^{-\frac{d}{2}}\log\left(\frac{1}{\lambda}\right)}$	$\left(\frac{\log x}{\log\log x}\right)^{-\frac{2}{d}}$
$\mathbb{Z}\wr(\ldots(\mathbb{Z}\wr(\mathbb{Z}\wr\mathbb{Z}))\ldots)$ k-times iterated wreath product, $k\geqslant 2$.	$e^{-t\left(\frac{\log_{(k)} t}{\log_{(k-1)} t}\right)^{2}}$	$e^{-\exp_{(k-1)}\left(\lambda^{-\frac{1}{2}}\log\left(\frac{1}{\lambda}\right)\right)}$	$\left(\frac{\log_{(k)}(x)}{\log_{(k+1)}(x)}\right)^{-2}$
$F\wr(\ldots(F\wr(F\wr\mathbb{Z}))\ldots)$ $\lvert F\rvert<\infty$, k times iterated wreath product $k\geqslant 2$	$e^{\frac{-t}{(\log_{(k-1)}(t))^{2}}}$	$e^{-\exp_{(k-1)}\left(\lambda^{-\frac{1}{2}}\right)}$	$\left(\log_{(k)}(x)\right)^{-2}$

SELF-SIMILARITY and actions on rooted trees

$$\underbrace{S_p \wr \cdots \wr S_p}_{n} \simeq \mathrm{Aut}\, T_{p,n}$$

- p-regular tree of depth n

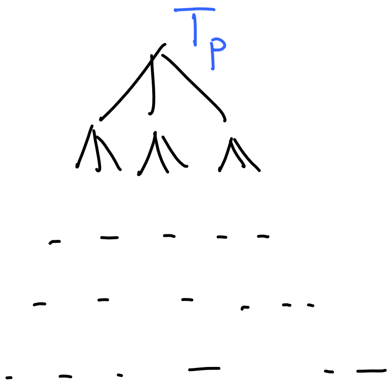


$$\underbrace{\mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p}_{n} \hookrightarrow \mathrm{Aut}\, T_{p,n}$$

Sylow p-subgroup

$$S_p \wr S_p \wr \dots \wr S_p \wr \dots \simeq \mathrm{Aut}\, T_p$$

infinite iterated wreath product



$$G \leq \mathrm{Aut}\, T$$

$\partial T = \{0, 1, \dots, p-1\}^{\mathbb{N}}$ - boundary

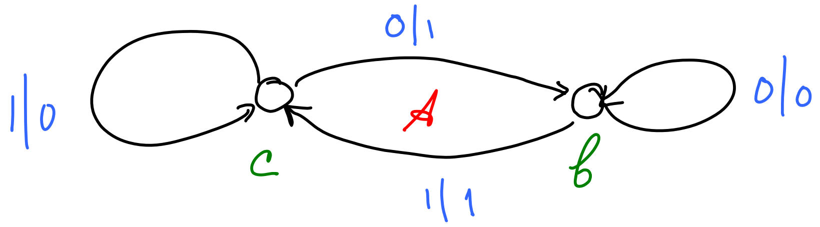
$$g = (g_0, \dots, g_{p-1})\sigma, \quad \sigma \in S_p$$

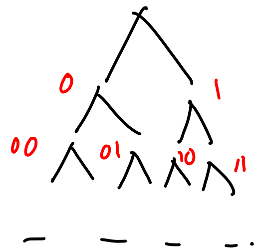
sections

G is self-similar if $\forall g$, $g_i \in G$ (after identification of T_i with T) $i = 0, \dots, p-1$.

$\Leftrightarrow$ $G = G(\mathcal{A})$, $\mathcal{A}$ - automaton of Mealey type.

Example.





Binary tree

Zuk and GRi 2001

1) $G(\mathcal{A}) \simeq \mathcal{L}$ under the map $a \to b^{-1}c,\ b \to b$.

2) The action of $\mathcal{L}$ given by automaton $\mathcal{L}$ on the boundary ∂T of binary tree is <u>essentially free</u>. [in contrast to BRANCH actions which are extremely non-free, like for $G=\langle a,b,c,d\rangle$]

3) The subgroup $St_{\mathcal{L}}(1)$ of index 2 in $\mathcal{L}$ is isomorphic to $\mathcal{L}$
(a counterexample to Benjamini conjecture raised 5 years later)

$\Rightarrow$ $\mathcal{L}$ is scale invariant

(later was developed by V. Nekrashevych and G. Pete)

4) The Markov operator $M = \lambda\left(\frac{a+a^{-1}+b+b^{-1}}{4}\right)$ has a pure point spectrum:

eigenvalues are $\cos\frac{p}{q}\pi$, $q \in \mathbb{Z}$, $(p,q)=1$, $0 \le p \le q-1$

and $\dim_{vN} \operatorname{Ker}\left(M - \cos\frac{p}{q}\pi\, I\right) = \frac{1}{2^q-1}$ = mass of the spectral measure at point $\cos\frac{p}{q}\pi$

↑ von Neumann dim

[This was used later to give a counter-example to the strong Atiyah Conjecture, discussed later]

GRi 2011 The essential freeness of the action of some self-similar groups on ∂T and the recurrent trace were used to construct asymptotic expanders.

The Atiyah Problem and the Lamplighter.

M. Atiyah 1976 $\qquad (M, g)$ - closed Riemannian manifold

$\tilde{M}$ - universal covering

$b^p_{(2)}(M, g)$ - L^2-Betti number (measure the size of the space of harmonic square-integrable p-forms on $\tilde{M}$)

Apriori, $b^p_{(2)}(M) \in \mathbb{R}$ but

$$\underset{\substack{| \\ \text{integer}}}{\chi(M)} = \sum_{p=0}^{\infty} (-1)^p \, b^p_{(2)}(M)$$

Euler characteristic $\rightarrow \chi(M)$

$$b^p_{(2)}(M) = b^p_{(2)}(\tilde{M}, \pi_1(M))$$

The Atiyah Problem. "A priori the numbers $b^p_{(2)}(M)$ are real. Give examples where they are not integral and even perhaps irrational."

The problem was converted into a bunch of Conjectures for manifolds and for groups.

G - countable group, $\mathcal{N}(G)$ - von Neumann algebra

tr_{vN} - von Neumann trace, $tr_{vN}(T) = \langle T\delta_e, \delta_e \rangle$

$\mathbb{C}[G]$, $\mathbb{R}[G]$, $\mathbb{Q}[G]$ - group rings over $\mathbb{C}, \mathbb{R}, \mathbb{Q}$

$A \in \mathbb{C}[G]$ - symmetric element $\longrightarrow \lambda_A$ right convolution operator (self-adjoin oper.)

We are interested in

$$\dim_{vN} \operatorname{Ker} \lambda_A = \operatorname{tr}_{vN}(P_A)$$

$P_A : \ell^2(G) \to \ell^2(G)$ is the orthogonal projection onto $\operatorname{Ker} \lambda_A$

Problem (The Atiyah problem for a group G). What is the set of $\dim \operatorname{Ker} \lambda_A$ when $A \in \mathbb{Q}[G]$ (or $\mathbb{Z}[G]$)?

if G is finitely presented and $\theta = \dim(\operatorname{Ker} \lambda_A)$, for $A \in \mathbb{Z}[G]$ then there exists a closed manifold M, with $\pi_1(M) \simeq G$ and such that one of the L^2-Betti numbers of M is equal to θ.

if G is recursively presented then G embeds into a finitely presented group H and $A \in \mathbb{Z}[G]$ becomes the element of $\mathbb{Z}[H]$ and $\dim_G(\operatorname{Ker}\lambda_A) = \dim_H(\operatorname{Ker}\lambda_A)$

$$@(G) = \{\dim(\operatorname{Ker}\lambda_A) : A \in \mathbb{Q}[G]\}$$ – ℓ^2-complexity of G.

A. Zuk and GRi 2001

Using the realization of the Lamplighter $\mathcal{L} = \mathbb{Z}_2 \wr \mathbb{Z}$ as automaton group, (essential) freenes of action of $\mathcal{L}$ on the boundary ∂T of binary tree and (implicitely) the recurrent trace τ on C_π (C^*-algebra generated by Koopman representation) showed that $\frac{1}{3} \in \mathcal{C}(\mathcal{L})$.

(1) This was used by P. Linnell, T. Schick, A. Zuk and GRi to construct a 7-dimensional smooth oriented Riemannian manifold (M,g) with $b^3_{(2)}(M) = \frac{1}{3}$ and
$\pi_1(M) \simeq \langle a, b, s \mid a^2 = [b,s] = [b^{-1}ab, a] = 1,\ s^{-1}as = [a,b] \rangle$

Baumslav-Remeslennikov group (ascending HNN-extension of $\mathcal{L}$).

[A counter-example to a strong version of the Atiyah Conj.]

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(2) T. Austin 2009. For some left-invariant subspace $V \subset \bigoplus_{F_2} \mathbb{Z}_2$, the finitely generated group $\left(\bigoplus_{F_2} \mathbb{Z}_2\right)/V \rtimes F_2$ admits a group ring element with rational coefficients

whose kernel has irrational (and even transcendental) von Neumann dimension.

what about recursively presented examples?

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3 L. Grabowski April 2010 1) The set of von Neumann dimensions arising from finitely generated groups is precisely the set of non-negative real numbers.

2) The set of von Neumann dimensions arising from finitely presented groups contains all numbers with recursive binary expansions.

3) Let $G = \langle a, b, s \rangle = \mathcal{L}(\mathbb{Q})_{\alpha}$ be Baumslag-Remeslennikov gp and $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the following group gives rise to the irrational von Neumann dimension

$$G \times G \times G \times (\hat{S} \rtimes \mathrm{Aut}(S)),$$

where the semidirect product is taken with respect to the natural action of $\mathrm{Aut}(S)$ on the Pontryagin dual $\hat{S}$.

4 M. Pichot, T. Schick, A. Zuk May 7 2010 showed that $\mathcal{C}(G)$ contains irrational number for a group G of type

$$\left(\bigoplus_{\Gamma} \mathbb{Z}_2 / V \right) \rtimes \Gamma$$

where V is a suitable Γ-invariant subspace of $\bigoplus_{\Gamma} \mathbb{Z}_2$ and Γ is either F_2 or $\mathbb{Z} \wr \mathbb{Z}$.

(5) F. Lehner and S. Wagner, May 13 2010: $\mathcal{C}(\mathbb{Z}_m \wr F_d)$, $d \geq 2$, $m \geq 2$.
contains an irrational algebraic number.

(6) L. Grabowski, September 1 2010 $\mathcal{C}(\mathbb{Z}_m \wr \mathbb{Z})$ contains transcendental numbers. $(m \geq 2)$

Two questions of Grabowski.

Q1. What is $\mathcal{C}(\mathbb{Z}_2 \wr \mathbb{Z})$?

Q2. is it the case that $\mathcal{C}(G) \not\subset \mathbb{Q}$ is equivalent to $\mathbb{Z}_m \wr \mathbb{Z} \hookrightarrow G$ for some m?

Problem. What is the spectral measure of Laplace operator on $\mathcal{L}$ for the standard system of generators a, b? Does it has a continious component?