Fixed points, Ramsey theorems, concentration of measure, and submeasures

Sławomir Solecki

May 2011

Outline of Topics

1. Extreme amenability
2. Groups of interest
3. Closed subgroups of S_∞
4. L_0 groups

Extreme amenability

Definition

A topological group G is **extremely amenable** if each continuous action of G on a compact space has a fixed point.

Definition

A topological group G is **extremely amenable** if each continuous action of G on a compact space has a fixed point.

Recall: G is amenable if each continuous action of G on a compact space has an invariant non-zero probability measure.

Veech: No locally compact group is extremely amenable.

Veech: No locally compact group is extremely amenable.

Gromov–Milman: Let H be a separable, infinite dimensional, complex Hilbert space. Let $U(H)$ be the unitary group with the strong operator topology. The group $U(H)$ is extremely amenable.

Veech: No locally compact group is extremely amenable.

Gromov–Milman: Let H be a separable, infinite dimensional, complex Hilbert space. Let $U(H)$ be the unitary group with the strong operator topology. The group $U(H)$ is extremely amenable.

We consider **Polish groups**, that is, completely metrizable, separable topological groups, for example, $U(H)$.

Groups of interest

Closed subgroups of S_∞

S_∞ the group of all permutations of $\mathbb{N}$ with the pointwise convergence topology.

S_∞ the group of all permutations of $\mathbb{N}$ with the pointwise convergence topology.

We consider **closed** $G < S_\infty$.

S_∞ the group of all permutations of $\mathbb{N}$ with the pointwise convergence topology.

We consider **closed** $G < S_\infty$.

These are precisely Polish groups with a basis at 1 consisting of open subgroups.

S_∞ the group of all permutations of $\mathbb{N}$ with the pointwise convergence topology.

We consider **closed** $G < S_\infty$.

These are precisely Polish groups with a basis at 1 consisting of open subgroups. They are totally disconnected.

L_0 groups

Let ϕ be a **submeasure** defined on clopen subsets $2^{\mathbb{N}}$,

Let ϕ be a **submeasure** defined on clopen subsets $2^{\mathbb{N}}$, so it is monotone, subadditive, and assigns 0 to $\emptyset$.

Let ϕ be a **submeasure** defined on clopen subsets $2^{\mathbb{N}}$, so it is monotone, subadditive, and assigns 0 to $\emptyset$.

ϕ is a **measure** if it is additive on disjoint sets.

Let ϕ be a **submeasure** defined on clopen subsets $2^{\mathbb{N}}$, so it is monotone, subadditive, and assigns 0 to $\emptyset$.

ϕ is a **measure** if it is additive on disjoint sets.

ϕ is **diffuse** if for each $\epsilon > 0$ there is a cover of $2^{\mathbb{N}}$ with clopen sets of ϕ-submeasure $< \epsilon$.

Let ϕ be a **submeasure** defined on clopen subsets $2^{\mathbb{N}}$, so it is monotone, subadditive, and assigns 0 to $\emptyset$.

ϕ is a **measure** if it is additive on disjoint sets.

ϕ is **diffuse** if for each $\epsilon > 0$ there is a cover of $2^{\mathbb{N}}$ with clopen sets of ϕ-submeasure $< \epsilon$.

We will assume that **submeasures are diffuse**.

We consider

$$L_0(\phi, H)$$

for a locally compact, second countable, **abelian** H.

We consider

$$L_0(\phi, H)$$

for a locally compact, second countable, **abelian** H.

Description: take the set S of all step-functions from $2^{\mathbb{N}}$ to the group (these are continuous functions with finitely many values); complete S with respect to convergence in submeasure ϕ.

We consider

$$L_0(\phi, H)$$

for a locally compact, second countable, **abelian** H.

Description: take the set S of all step-functions from $2^{\mathbb{N}}$ to the group (these are continuous functions with finitely many values); complete S with respect to convergence in submeasure ϕ.

These groups are connected.

We consider

$$L_0(\phi, H)$$

for a locally compact, second countable, **abelian** H.

Description: take the set S of all step-functions from $2^{\mathbb{N}}$ to the group (these are continuous functions with finitely many values); complete S with respect to convergence in submeasure ϕ.

These groups are connected. If ϕ is a measure, then $L_0(\phi, \mathbb{T})$ shows up as a maximal abelian subgroup of $U(H)$.

We consider

$$L_0(\phi, H)$$

for a locally compact, second countable, **abelian** H.

Description: take the set S of all step-functions from $2^{\mathbb{N}}$ to the group (these are continuous functions with finitely many values); complete S with respect to convergence in submeasure ϕ.

These groups are connected. If ϕ is a measure, then $L_0(\phi, \mathbb{T})$ shows up as a maximal abelian subgroup of $U(H)$.

$$L_0(\phi, \mathbb{Z}/2),\ L_0(\phi, \mathbb{Z}),\ L_0(\phi, \mathbb{T}),\ L_0(\phi, \mathbb{R})$$

is a representative sample.

Closed subgroups of S_∞

Consider a countable family $\mathcal{F}$ of finite structures.

Consider a countable family $\mathcal{F}$ of finite structures.

Examples:

Consider a countable family $\mathcal{F}$ of finite structures.

Examples:

1. finite linear orders;

Consider a countable family $\mathcal{F}$ of finite structures.

Examples:

1. finite linear orders;
2. finite graphs/finite linearly ordered graphs;

Consider a countable family $\mathcal{F}$ of finite structures.

Examples:

1. finite linear orders;
2. finite graphs/finite linearly ordered graphs;
3. finite metric spaces with rational distances/linearly ordered such metric spaces.

Assumptions on $\mathcal{F}$:

Assumptions on $\mathcal{F}$:

(a) closed under substructures;

Assumptions on $\mathcal{F}$:

(a) closed under substructures;

(b) directed with respect to embeddings;

Assumptions on $\mathcal{F}$:

(a) closed under substructures;
(b) directed with respect to embeddings;
(c) amalgamation holds in it.

Assumptions on $\mathcal{F}$:

(a) closed under substructures;
(b) directed with respect to embeddings;
(c) amalgamation holds in it.

Such an $\mathcal{F}$ is called a **Fraissé class**.

Assumptions on $\mathcal{F}$:

(a) closed under substructures;

(b) directed with respect to embeddings;

(c) amalgamation holds in it.

Such an $\mathcal{F}$ is called a **Fraissé class**.

All the classes mentioned above are Fraissé.

Given a countable Fraissé class $\mathcal{F}$, there is a unique ultrahomogeneous countable structure M with

$$\mathcal{F} = \text{ all finite substructures of } M.$$

Given a countable Fraissé class $\mathcal{F}$, there is a unique ultrahomogeneous countable structure M with

$$\mathcal{F} = \text{ all finite substructures of } M.$$

Ultrahomogeneous = all partial *finite* isomorphisms extend to automorphisms of M.

Given a countable Fraissé class $\mathcal{F}$, there is a unique ultrahomogeneous countable structure M with

$$\mathcal{F} = \text{ all finite substructures of } M.$$

Ultrahomogeneous = all partial *finite* isomorphisms extend to automorphisms of M.

Notation:

$$M = \lim \mathcal{F}.$$

Given a countable Fraissé class $\mathcal{F}$, there is a unique ultrahomogeneous countable structure M with

$$\mathcal{F} = \text{ all finite substructures of } M.$$

Ultrahomogeneous = all partial *finite* isomorphisms extend to automorphisms of M.

Notation:

$$M = \lim \mathcal{F}.$$

M is the **Fraissé limit of** $\mathcal{F}$.

Examples:

Examples:

1. $\mathbb{Q} = \lim\{\text{finite linear orders}\}$;

Examples:

1. $\mathbb{Q} = \lim\{\text{finite linear orders}\}$;
2. random graph $= \lim\{\text{finite graphs}\}$;

Examples:

1. $\mathbb{Q} = \lim\{$finite linear orders$\}$;
2. random graph $= \lim\{$finite graphs$\}$;
   linearly ordered random graph $=$
   $\lim\{$linearly ordered finite graphs$\}$;

Examples:

1. $\mathbb{Q} = \lim\{$finite linear orders$\}$;
2. random graph $= \lim\{$finite graphs$\}$;
   linearly ordered random graph $=$
   $\lim\{$linearly ordered finite graphs$\}$;
3. rational Urysohn space $=$
   $\lim\{$finite metric spaces with rational distances$\}$;

Examples:

1. $\mathbb{Q} = \lim\{\text{finite linear orders}\}$;
2. random graph $= \lim\{\text{finite graphs}\}$;
   linearly ordered random graph $=$
   $\lim\{\text{linearly ordered finite graphs}\}$;
3. rational Urysohn space $=$
   $\lim\{\text{finite metric spaces with rational distances}\}$;
   linearly ordered rational Urysohn space $=$
   $\lim\{\text{linearly ordedred finite metric spaces with rational distances}\}$.

G an **arbitrary closed** subgroup of S_∞.

G an **arbitrary closed** subgroup of S_∞. There is a countable Fraissé class $\mathcal{F}$ with

$$G = \mathrm{Aut}(\lim \mathcal{F}).$$

G an **arbitrary closed** subgroup of S_∞. There is a countable Fraissé class $\mathcal{F}$ with

$$G = \mathrm{Aut}(\lim \mathcal{F}).$$

So $\mathrm{Aut}(\lim \mathcal{F})$ is a completely general closed subgroup of S_∞.

$\mathcal{F}$ has the **Ramsey property**

$\mathcal{F}$ has the **Ramsey property** if for each $A, B \in \mathcal{F}$ and an integer $d > 0$

$\mathcal{F}$ has the **Ramsey property** if for each $A, B \in \mathcal{F}$ and an integer $d > 0$ there exists $C \in \mathcal{F}$ such that for each d-coloring of all copies of A in C

$\mathcal{F}$ has the **Ramsey property** if for each $A, B \in \mathcal{F}$ and an integer $d > 0$ there exists $C \in \mathcal{F}$ such that for each d-coloring of all copies of A in C there is a copy B' of B in C such that all copies of A in B' get the same color.

$\mathcal{F}$ has the **Ramsey property** if for each $A, B \in \mathcal{F}$ and an integer $d > 0$ there exists $C \in \mathcal{F}$ such that for each d-coloring of all copies of A in C there is a copy B' of B in C such that all copies of A in B' get the same color.

$\mathcal{F}$ is **ordered**

$\mathcal{F}$ has the **Ramsey property** if for each $A, B \in \mathcal{F}$ and an integer $d > 0$ there exists $C \in \mathcal{F}$ such that for each d-coloring of all copies of A in C there is a copy B' of B in C such that all copies of A in B' get the same color.

$\mathcal{F}$ is **ordered** if each structure in $\mathcal{F}$ carries a linear order.

Theorem (Kechris–Pestov–Todorcevic)

Let $\mathcal{F}$ be a countable Fraissé class. Assume that $\mathcal{F}$ is ordered *and has the* Ramsey property*. Then $\mathrm{Aut}(\lim \mathcal{F})$ is extremely amenable.*

Theorem (Kechris–Pestov–Todorcevic)

Let $\mathcal{F}$ be a countable Fraissé class. Assume that $\mathcal{F}$ is ordered *and has the* Ramsey property*. Then $\mathrm{Aut}(\lim \mathcal{F})$ is extremely amenable. The converse holds as well.*

Connections with **Structural Ramsey Theory**:

Connections with **Structural Ramsey Theory**:

Ramsey: finite linear orders have the Ramsey property;

Connections with **Structural Ramsey Theory**:

Ramsey: finite linear orders have the Ramsey property;

Nešetřil–Rödl: finite linearly ordered graphs have the Ramsey property;

Connections with **Structural Ramsey Theory**:

Ramsey: finite linear orders have the Ramsey property;

Nešetřil–Rödl: finite linearly ordered graphs have the Ramsey property;

Nešetřil: finite linearly ordered metric spaces with rational distances have the Ramsey property.

Connections with **Structural Ramsey Theory**:

Ramsey: finite linear orders have the Ramsey property;

Nešetřil–Rödl: finite linearly ordered graphs have the Ramsey property;

Nešetřil: finite linearly ordered metric spaces with rational distances have the Ramsey property.

Consequences:

Connections with **Structural Ramsey Theory**:

Ramsey: finite linear orders have the Ramsey property;

Nešetřil–Rödl: finite linearly ordered graphs have the Ramsey property;

Nešetřil: finite linearly ordered metric spaces with rational distances have the Ramsey property.

Consequences:

$\mathrm{Aut}(\mathbb{Q})$, Aut(linearly ordered random graph),
Aut(linearly ordered rational Urysohn) are all extremely amenable.

Connections with **Structural Ramsey Theory**:

Ramsey: finite linear orders have the Ramsey property;

Nešetřil–Rödl: finite linearly ordered graphs have the Ramsey property;

Nešetřil: finite linearly ordered metric spaces with rational distances have the Ramsey property.

Consequences:

$\mathrm{Aut}(\mathbb{Q})$, Aut(linearly ordered random graph), Aut(linearly ordered rational Urysohn) are all extremely amenable.

The results above make it possible to compute universal minimal flows of Aut(random graph) and Aut(rational Urysohn).

L_0 groups

Conventions:

Conventions:

– ϕ is a diffuse submeasure;

Conventions:

- ϕ is a diffuse submeasure;
- H is a locally compact, second countable, **abelian** group;

Conventions:

- ϕ is a diffuse submeasure;
- H is a locally compact, second countable, **abelian** group;
- consider $L_0(\phi, H)$.

Theorem (Farah–S.)

$L_0(\phi, H)$ is extremely amenable if H is compact.

Theorem (Farah–S.)

$L_0(\phi, H)$ is extremely amenable if H is compact.

The proof uses a Ramsey–type theorem,

Theorem (Farah–S.)

$L_0(\phi, H)$ *is extremely amenable if* H *is compact.*

The proof uses a Ramsey–type theorem, whose proof relies on a generalization of the Borsuk–Ulam theorem on antipodal points.

Types of submeasures ϕ:

Types of submeasures ϕ:

– pathological submeasures (no measures $\leq \phi$);

Types of submeasures ϕ:

- pathological submeasures (no measures $\leq \phi$);
- measures;

Types of submeasures ϕ:

– pathological submeasures (no measures $\leq \phi$);

– measures;

– Hausdorff submeasures (many measures $\leq \phi$).

For a sequence of subsets $A_1, \ldots, A_n$ of X, let

$$t(A_1, \ldots, A_n) = \max\{k\colon \forall x \in X \; |\{i \leq n\colon x \in A_i\}| \geq k\}.$$

For a sequence of subsets $A_1, \ldots, A_n$ of X, let

$$t(A_1, \ldots, A_n) = \max\{k\colon \forall x \in X\ |\{i \leq n\colon x \in A_i\}| \geq k\}.$$

ϕ a submeasure.

For a sequence of subsets $A_1, \ldots, A_n$ of X, let

$$t(A_1, \ldots, A_n) = \max\{k \colon \forall x \in X \ |\{i \leq n \colon x \in A_i\}| \geq k\}.$$

ϕ a submeasure. Let $\mathcal{A}_\xi$ consist of all clopen sets with $\phi(A) < \xi$.

For a sequence of subsets $A_1, \ldots, A_n$ of X, let

$$t(A_1, \ldots, A_n) = \max\{k \colon \forall x \in X\ |\{i \leq n \colon x \in A_i\}| \geq k\}.$$

ϕ a submeasure. Let $\mathcal{A}_\xi$ consist of all clopen sets with $\phi(A) < \xi$.

Define

$$h_\phi(\xi) = \sup \frac{t(A_1, \ldots, A_n)}{\xi \cdot n},$$

where sup is taken over all sequences $A_1, \ldots, A_n$ of sets in $\mathcal{A}_\xi$.

Proposition (S.)

Let ϕ be a submeasure. Then

$$h_\phi \sim \frac{1}{\xi}$$

or

$$h_\phi \sim a \ \text{ for some } 0 \leq a < \infty.$$

Proposition (S.)

Let ϕ be a submeasure. Then

$$h_\phi \sim \frac{1}{\xi}$$

or

$$h_\phi \sim a \ \text{ for some } 0 \leq a < \infty.$$

ϕ pathological $\Leftrightarrow h_\phi \sim \frac{1}{\xi}$.

Proposition (S.)

Let ϕ be a submeasure. Then

$$h_\phi \sim \frac{1}{\xi}$$

or

$$h_\phi \sim a \ \text{ for some } 0 \leq a < \infty.$$

ϕ pathological $\Leftrightarrow$ $h_\phi \sim \frac{1}{\xi}$.

ϕ a measure $\Rightarrow$ $h_\phi \sim 1$.

Proposition (S.)

Let ϕ be a submeasure. Then

$$h_\phi \sim \frac{1}{\xi}$$

or

$$h_\phi \sim a \ \text{ for some } 0 \leq a < \infty.$$

ϕ pathological $\Leftrightarrow h_\phi \sim \frac{1}{\xi}$.

ϕ a measure $\Rightarrow h_\phi \sim 1$.

ϕ Hausdorff $\Leftrightarrow h_\phi \sim 0$.

Pathological submeasures are generic submeasures.

Pathological submeasures are generic submeasures. They show up, for example, in measure theory, in graph theory, and in the study of ideals of subsets of $\mathbb{N}$.

Pathological submeasures are generic submeasures. They show up, for example, in measure theory, in graph theory, and in the study of ideals of subsets of $\mathbb{N}$.

Hausdorff submeasures show up as approximations to Hausdorff measures.

Pathological submeasures

Pathological submeasures

Theorem (Herer–Christensen, S.)

$L_0(\phi, H)$ is extremely amenable if ϕ is pathological.

Pathological submeasures

Theorem (Herer–Christensen, S.)

$L_0(\phi, H)$ is extremely amenable if ϕ is pathological.

Herer–Christensen's proof for $H = \mathbb{R}$ uses spectral theorem.

Pathological submeasures

Theorem (Herer–Christensen, S.)

$L_0(\phi, H)$ *is extremely amenable if* ϕ *is pathological.*

Herer–Christensen's proof for $H = \mathbb{R}$ uses spectral theorem.

S's proof for general H uses **concentration of measure** derived from the Brascamp–Lieb inequality,

Pathological submeasures

Theorem (Herer–Christensen, S.)

$L_0(\phi, H)$ is extremely amenable if ϕ is pathological.

Herer–Christensen's proof for $H = \mathbb{R}$ uses spectral theorem.

S's proof for general H uses **concentration of measure** derived from the Brascamp–Lieb inequality, which generalizes the Loomis–Whitney inequality

$$\mathrm{vol}(A) \leq \prod_{i=1}^{n} \mathrm{vol}(\mathrm{proj}_i(A))^{1/n-1},$$

where $A \subseteq \mathbb{R}^n$ and proj_i is the projection along the i-th axis.

Measures

Measures

Theorem (Glasner, Furstenberg–Weiss)

$L_0(\phi, H)$ is extremely amenable if ϕ is a measure.

Measures

Theorem (Glasner, Furstenberg–Weiss)

$L_0(\phi, H)$ is extremely amenable if ϕ is a measure.

The proof uses the classical **concentration of measure** in product spaces going back to Milman–Schechtman.

Concentration of measure enters the proofs through the following notion.

Concentration of measure enters the proofs through the following notion.

Gromov–Milman: G is **Lévy** if

Concentration of measure enters the proofs through the following notion.

Gromov–Milman: G is **Lévy** if there are compact subgroups $K_1 < K_2 < K_3 < \cdots$ such that

Concentration of measure enters the proofs through the following notion.

Gromov–Milman: G is **Lévy** if there are compact subgroups $K_1 < K_2 < K_3 < \cdots$ such that $\bigcup_n K_n$ is dense in G and

Concentration of measure enters the proofs through the following notion.

Gromov–Milman: G is **Lévy** if there are compact subgroups $K_1 < K_2 < K_3 < \cdots$ such that $\bigcup_n K_n$ is dense in G and for all $A_n \subset K_n$ with $\inf_n \mu_n(A_n) > 0$ and for each $U \ni 1$ open in G, we have

$$\lim_n \mu_n(UA_n) = 1,$$

where μ_n is the normalized Haar measure on K_n.

Hausdorff submeasures

Hausdorff submeasures

Theorem (Farah–S.)

$L_0(\phi, H)$ does **not** *have concentration of measure (is not Lévy) for zero dimensional, compact, non-trivial H and for certain Hausdorff ϕ.*

Question: Is $L_0(\phi, \mathbb{Z})$ extremely amenable for some (Hausdorff) ϕ?

Question: Is $L_0(\phi, \mathbb{Z})$ extremely amenable for some (Hausdorff) ϕ?

Hope: NO.

Question: Is $L_0(\phi, \mathbb{Z})$ extremely amenable for some (Hausdorff) ϕ?

Hope: NO.

It would produce an abelian group with no characters but with a fixed point free, continuous action on a compact space, answering a question of Pestov.

Question: Is $L_0(\phi, H)$ extremely amenable for each compact, not necessarily abelian, group H?

Question: Is $L_0(\phi, H)$ extremely amenable for each compact, not necessarily abelian, group H?

An extension of the Ramsey theorem based on Borsuk–Ulam to non-commutative context is needed.